

TIT/HEP-345  
 UT-Komaba 96-18  
 hep-th/9611041  
 November 1996

# String Theory on Three Dimensional Black Holes

MAKOTO NATSUUME <sup>†</sup>

*Department of Physics, Tokyo Institute of Technology  
 Oh-Okayama, Meguro, Tokyo 152, Japan*

and

YUJI SATOH <sup>‡</sup>

*Institute of Physics, University of Tokyo, Komaba  
 Meguro, Tokyo 153, Japan*

## Abstract

We investigate the string theory on three dimensional black holes discovered by Bañados, Teitelboim and Zanelli in the framework of conformal field theory. The model is described by an orbifold of the  $\widetilde{SL}(2, \mathbf{R})$  WZW model. The spectrum is analyzed by solving the level matching condition and we obtain winding modes. We then study the ghost problem and show explicit examples of physical states with negative norms. We discuss the tachyon propagation and the target space geometry, which are irrelevant to the details of the spectrum. We find a self-dual T-duality transformation reversing the black hole mass. We also discuss difficulties in string theory on curved spacetime and possibilities to obtain a sensible string theory on three dimensional black holes. This work is the first attempt to quantize a string theory in a black hole background with an infinite number of propagating modes.

PACS codes: 04.70.Dy, 11.25.-w, 11.25.Hf, 11.40 Ex

Keywords: BTZ black holes, WZW model, Orbifold, Unitarity

---

<sup>†</sup>makoto@th.phys.titech.ac.jp; JSPS Research Fellow

<sup>‡</sup>ysatoh@hep1.c.u-tokyo.ac.jp; JSPS Research Fellow

# 1 Introduction

Black holes provide useful laboratories in quantum gravity. Through the study of black holes, we expect to obtain useful insights in order to solve problems such as singularities, black hole thermodynamics and Hawking radiation. In string theory, most discussions on black hole physics are based on low energy effective theories, but for definite arguments we have to develop analysis beyond the  $\alpha'$  expansion.

Many works have been devoted to the  $SL(2, \mathbf{R})/U(1)$  black hole [1, 2] for that reason. However, most works are based on the semi-classical analysis, e.g., [1]-[3] and we need further investigations in order to clarify important issues in black hole physics.<sup>1</sup> The difficulties are rooted in the fact that the target space is non-compact and curved in time direction.

Such difficulties are not characteristic of string theories in black hole backgrounds. In general, as a sensible physical theory, a string theory has to satisfy various consistency conditions. Although we have many consistent string theories on curved spaces, i.e., on group manifolds, they are compact and must be tensored with Minkowski spacetime. We have few consistent string theories with curved time. For instance, the no-ghost theorem requires a flat light-cone direction. Even though most proofs [8] are stated for the  $D = 26$  bosonic string, many can be extended easily to the general  $c = 26$  matter CFT with  $D$  dimensional Minkowski spacetime and a compact CFT. The only assumption needed for the compact CFT is that it is conformally invariant with the appropriate central charge so that there is a nilpotent BRST operator, and that it has a positive inner product. However, all known proofs require  $D$  to be at least two. There is no general result for  $D < 2$ .

Since string theory is regarded as the fundamental theory including gravity, it is important to construct a consistent string theory on curved *spacetime*. There have been a few previous attempts besides the  $SL(2, \mathbf{R})/U(1)$  case. For example, there are various attempts using the  $SL(2, \mathbf{R})$  WZW model [9]-[15], but it is known to contain ghosts.<sup>2</sup> Russo and Tseytlin has discussed a string theory in a curved background which can be transformed to a flat theory by T-duality [17].

The purpose of this paper is to formulate the string theory on the three dimensional black hole discovered by Bañados, Teitelboim and Zanelli (BTZ) [18]. This black hole is important in string theory. This is one of few known exact solutions in string theory and one of the simplest solutions; the solution is described by an orbifold of the  $\widetilde{SL}(2, \mathbf{R})$  WZW model [19, 20]. Moreover, strings in three dimensions have an infinite number of propagating modes, so it resembles higher dimensional ones.

The BTZ black hole provides a background to the bosonic string, but it was originally

---

<sup>1</sup>See however [4]-[7] for example.

<sup>2</sup>A resolution to the ghost problem has been proposed though [16].

found as a solution to Einstein gravity. In fact, it has been extensively studied in Einstein gravity (for a review, see [21]). The BTZ black hole shares many properties with the  $(3+1)$ -dimensional black hole. But it is simpler since it is locally three dimensional anti-de Sitter ( $AdS_3$ ) space. This simplicity enables us to investigate many characteristics of the black hole physics in an explicit manner without mathematical complications. In the classical theory, for example, the gravitational collapse and the instability of the inner horizon have been studied in detail. Quantum field theory on the BTZ black hole has been also explored and exact results are known about Green functions, mode functions and thermodynamic quantities of scalar fields. Furthermore, its thermodynamic and statistical mechanical properties have been investigated by the Chern-Simons formulation of the  $(2+1)$ -dimensional general relativity.

Therefore, the subject is important both as a quantum black hole and as a string theory in a nontrivial background. Nevertheless, the detailed construction of the orbifold has not been made so far. In this paper, we will investigate the spectrum of the theory, the ghost problem, the tachyon propagation and the target space geometry. Although we cannot overcome all the problems, this work may provide useful insights into these issues. Besides this work is the first attempt to quantize a string theory in a black hole background with an infinite number of propagating modes.

The organization of the present paper is as follows. In Sec. 2, we briefly review the BTZ black hole using the  $\widetilde{SL}(2, \mathbf{R})$  WZW model. In Sec. 3, we develop the conformal field theory for the three dimensional black hole. We investigate the spectrum by solving the level matching condition. Then we investigate the issue of ghosts in Sec. 4. We find explicit examples of physical states with negative norms. In Sec. 5, we study the tachyon propagation and the target space geometry. We discuss states localized near the black hole and discuss a T-duality transformation reversing the black hole mass. In Sec. 6, we discuss the other consistency conditions in string theory and discuss difficulties in the case of curved spacetime. Basic properties of the representation theory of  $\widetilde{SL}(2, \mathbf{R})$ , which are necessary in the text, are summarized in Appendix A. Representations in the hyperbolic basis are explained in some detail. Also, we show the Clebsch-Gordan decomposition of the  $sl(2, \mathbf{R})$  Kac-Moody module in the hyperbolic basis in Appendix B.

## 2 The $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$ black hole

In this section, we briefly review how to describe the BTZ black hole [18] from the  $SL(2, \mathbf{R})$  WZW model [19, 20] and summarize basic facts on the  $SL(2, \mathbf{R})$  WZW model.

## 2.1 The BTZ black hole as a string background

We start with the  $SL(2, \mathbf{R})$  WZW model <sup>3</sup> with action

$$\frac{k}{8\pi} \int_{\Sigma} d^2\sigma \sqrt{h} h^{\alpha\beta} \text{Tr} \left( g^{-1} \partial_{\alpha} g g^{-1} \partial_{\beta} g \right) + ik \Gamma(g), \quad (2.1)$$

where  $h_{\alpha\beta}$  is the metric on a Riemann surface  $\Sigma$  and  $g$  is an element of  $SL(2, \mathbf{R})$ .  $\Gamma$  is the Wess-Zumino term given by

$$\frac{1}{12\pi} \int_B \text{Tr} \left( g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right), \quad (2.2)$$

where  $B$  is a three manifold with boundary  $\Sigma$ . We parametrize  $g$  by

$$g = \begin{pmatrix} x_1 + x_2 & x_3 + x_0 \\ x_3 - x_0 & x_1 - x_2 \end{pmatrix}, \quad (2.3)$$

$$\det g = x_0^2 + x_1^2 - x_2^2 - x_3^2 = 1. \quad (2.4)$$

The latter equation is nothing but the embedding equation of the three dimensional anti-de Sitter space ( $AdS_3$ ) in a flat space; thus  $SL(2, \mathbf{R})$  and  $AdS_3$  are the same manifold. This is the reason why the BTZ black hole is described by the  $SL(2, \mathbf{R})$  WZW model.

In order to unwrap the compact time direction of  $SL(2, \mathbf{R})$ , we go to the universal covering group  $\widetilde{SL}(2, \mathbf{R})$  and consider three regions parametrized by

$$\begin{aligned} \text{Region I} \quad (\hat{r}^2 > 1) & : \quad x_1 = \hat{r} \cosh \hat{\varphi}, & x_0 = \sqrt{\hat{r}^2 - 1} \sinh \hat{t}, \\ & \quad x_2 = \hat{r} \sinh \hat{\varphi}, & x_3 = \sqrt{\hat{r}^2 - 1} \cosh \hat{t}, \\ \text{Region II} \quad (1 > \hat{r}^2 > 0) & : \quad x_1 = \hat{r} \cosh \hat{\varphi}, & x_0 = \sqrt{1 - \hat{r}^2} \cosh \hat{t}, \\ & \quad x_2 = \hat{r} \sinh \hat{\varphi}, & x_3 = \sqrt{1 - \hat{r}^2} \sinh \hat{t}, \\ \text{Region III} \quad (0 > \hat{r}^2) & : \quad x_1 = \sqrt{-\hat{r}^2} \sinh \hat{\varphi}, & x_0 = \sqrt{1 - \hat{r}^2} \cosh \hat{t}, \\ & \quad x_2 = \sqrt{-\hat{r}^2} \cosh \hat{\varphi}, & x_3 = \sqrt{1 - \hat{r}^2} \sinh \hat{t}, \end{aligned} \quad (2.5)$$

where  $-\infty < \hat{t}, \hat{\varphi} < \infty$ . <sup>4</sup> These regions describe I) the region outside the outer horizon, II) the region between the outer and the inner horizon, and III) the region inside the inner horizon of the black hole. In every parametrization, the WZW action takes the form

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left( h^{\alpha\beta} G_{\mu\nu} + i\epsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu}, \quad (2.6)$$

where

$$\begin{aligned} ds^2 &= \alpha' k \left\{ -(\hat{r}^2 - 1) d\hat{t}^2 + \hat{r}^2 d\hat{\varphi}^2 + (\hat{r}^2 - 1)^{-1} d\hat{r}^2 \right\}, \\ B &= \alpha' k \hat{r}^2 d\hat{\varphi} \wedge d\hat{t}. \end{aligned} \quad (2.7)$$

---

<sup>3</sup>There are difficulties to construct a CFT based on a non-compact group manifold. In this paper, we will simply assume the existence of the  $\widetilde{SL}(2, \mathbf{R})$  WZW model.

<sup>4</sup>Throughout this paper, we use dimensionless coordinates. The dimension is recovered by the cosmological constant  $-l^{-2}$  if necessary.

We make a further change of variables:

$$\hat{r}^2 = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}, \quad \begin{pmatrix} \hat{t} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} t \\ \varphi \end{pmatrix}, \quad (2.8)$$

where  $r_{\pm}$  ( $r_+ > r_-$ ) are positive constants. Then, we get

$$\begin{aligned} ds^2 &= \alpha' k \left\{ -\left(r^2 - M_{BH}\right) dt^2 - J_{BH} dt d\varphi + r^2 d\varphi^2 + \left(r^2 - M_{BH} + \frac{J_{BH}^2}{4r^2}\right)^{-1} dr^2 \right\}, \\ B &= \alpha' k r^2 d\varphi \wedge dt, \end{aligned} \quad (2.9)$$

where  $M_{BH} = r_+^2 + r_-^2$  and  $J_{BH} = 2r_+r_-$ .  $B$  is defined up to an exact form. By identifying  $\varphi$  with  $\varphi + 2\pi$  and dropping the region  $r^2 < 0$ , we obtain the BTZ black hole. The coordinates in (2.9) now take  $-\infty < t < +\infty$ ,  $0 \leq \varphi < 2\pi$  and  $0 \leq r < +\infty$ .  $r_+$  and  $r_-$  represent the location of the outer and the inner horizon.  $M_{BH}$  and  $J_{BH}$  are the mass and the angular momentum of the black hole respectively.

The non-rotating black hole is obtained by  $r_- = 0$ . The extremal black hole is obtained by  $r_+ = r_-$  in (2.9) although various intermediate expressions become singular. One can show that the above geometry is a solution to low energy field equations. Moreover, the exact metric and anti-symmetric tensor are given by the replacement  $k$  with  $k - 2$  [22], where  $-2$  is the second Casimir of the adjoint representation of  $sl(2, \mathbf{R})$ . The cosmological constant is given by  $-l^{-2} = -\alpha'^{-1}(k - 2)^{-1}$ .

## 2.2 Chiral currents and the stress tensor

The  $\widetilde{SL}(2, \mathbf{R})$  WZW model has a chiral  $\widetilde{SL}(2, \mathbf{R})_L \times \widetilde{SL}(2, \mathbf{R})_R$  symmetry. The corresponding currents are given by

$$J(z) = \frac{ik}{2} \partial g g^{-1}, \quad \tilde{J}(\bar{z}) = \frac{ik}{2} g^{-1} \bar{\partial} g, \quad (2.10)$$

where  $z = e^{\tau+i\sigma}$  and  $\bar{z} = e^{\tau-i\sigma}$ . The currents act on  $g$  as

$$J^a(z)g(w, \bar{w}) \sim \frac{-\tau^a g}{z - w}, \quad \tilde{J}^a(\bar{z})g(w, \bar{w}) \sim \frac{-g \tau^a}{\bar{z} - \bar{w}}. \quad (2.11)$$

Here, we have defined  $J^a$  ( $a = 0, 1, 2$ ) by  $J(z) = \eta_{ab} \tau^a J^b(z)$  and similarly for  $\tilde{J}^a$ , where  $\eta_{ab} = \text{diag}(-1, 1, 1)$ .  $\tau^a$  form a basis of  $sl(2, \mathbf{R})$  with the properties

$$[\tau^a, \tau^b] = i\epsilon^a{}_b \tau^c, \quad \text{Tr}(\tau^a \tau^b) = -\frac{1}{2} \eta^{ab}. \quad (2.12)$$

In terms of the Pauli matrices,  $\tau^0 = -\sigma^2/2$ ,  $\tau^1 = i\sigma^1/2$  and  $\tau^2 = i\sigma^3/2$ . The stress tensor is given by

$$T(z) = \frac{1}{k-2} \eta_{ab} J^a(z) J^b(z). \quad (2.13)$$

The conformal modes of the currents and the stress tensor satisfy the commutation relations

$$\begin{aligned}
[J_n^a, J_m^b] &= i\epsilon^{ab}{}_c J_{n+m}^c + \frac{k}{2} n \eta^{ab} \delta_{n+m}, \\
[L_n, J_m^a] &= -m J_{n+m}^a, \\
[L_n, L_m] &= (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m},
\end{aligned} \tag{2.14}$$

where  $c = 3k/(k-2)$ . For the critical value  $c = 26$ , we have  $k = 52/23$ . The above Kac-Moody algebra is expressed in the basis  $I_n^\pm \equiv J_n^1 \pm iJ_n^2$  and  $I_n^0 \equiv J_n^0$  as

$$\begin{aligned}
[I_n^+, I_m^-] &= -2I_{n+m}^0 + kn\delta_{n+m}, & [I_n^\pm, I_m^\pm] &= 0, \\
[I_n^0, I_m^\pm] &= \pm I_{n+m}^\pm, & [I_n^0, I_m^0] &= -\frac{k}{2} n \delta_{n+m}.
\end{aligned} \tag{2.15}$$

On the other hand, in the basis  $J_n^\pm \equiv J_n^0 \pm J_n^1$  and  $J_n^2$ , the algebra is written as

$$\begin{aligned}
[J_n^+, J_m^-] &= -2iJ_{n+m}^2 - kn\delta_{n+m}, & [J_n^\pm, J_m^\pm] &= 0, \\
[J_n^2, J_m^\pm] &= \pm iJ_{n+m}^\pm, & [J_n^2, J_m^2] &= \frac{k}{2} n \delta_{n+m}.
\end{aligned} \tag{2.16}$$

In this paper, we will utilize the latter basis. Note the Hermite conjugates for the latter basis are given by

$$(J_m^\pm)^\dagger = J_{-m}^\pm, \quad (J_m^2)^\dagger = J_{-m}^2. \tag{2.17}$$

Similar expressions hold for the anti-holomorphic part.

## 2.3 Twisting

As explained in the previous subsection, in order to get the three dimensional black hole, we have (i) to go to the universal covering space of  $SL(2, \mathbf{R})$ , (ii) to make the identification  $\varphi \sim \varphi + 2\pi$  and (iii) to drop the region  $r^2 < 0$ . We can take (i) into account by considering the representation theory of  $\widetilde{SL}(2, \mathbf{R})$  instead of  $SL(2, \mathbf{R})$ . The point (iii) is related to the problem of closed timelike curves [18, 19]; we will discuss this point in Sec. 6. For now, we will concentrate on (ii).

From the  $AdS_3$  point of view, the translations of  $\hat{t}$  and  $\hat{\varphi}$  correspond to boosts in the flat spacetime in which  $AdS_3$  is embedded. From (2.8), the translation of  $\varphi$  is given by a linear combination of those of  $\hat{t}$  and  $\hat{\varphi}$ . In terms of the  $SL(2, \mathbf{R})$  WZW model, the translations of  $\hat{t}$  and  $\hat{\varphi}$  correspond to a vector and an axial symmetry of the WZW model [19]. If we gauge these symmetries, the resulting coset theories are the  $SL(2, \mathbf{R})/U(1)$  black holes [1].

In order to express these translations by the  $sl(2, \mathbf{R})$  currents, it is convenient to parametrize the group manifold by analogues of Euler angles; we parametrize Region I-III by

$$\begin{aligned}
\text{Region I} \quad : \quad g &= e^{-i\theta_L \tau^2} e^{-i\rho \tau^1} e^{-i\theta_R \tau^2} = \begin{pmatrix} e^{\hat{\varphi}} \cosh \rho/2 & e^{\hat{t}} \sinh \rho/2 \\ e^{-\hat{t}} \sinh \rho/2 & e^{-\hat{\varphi}} \cosh \rho/2 \end{pmatrix}, \\
\text{Region II} \quad : \quad g &= e^{-i\theta_L \tau^2} e^{-i\rho \tau^0} e^{-i\theta_R \tau^2} = \begin{pmatrix} e^{\hat{\varphi}} \cos \rho/2 & e^{\hat{t}} \sin \rho/2 \\ -e^{-\hat{t}} \sin \rho/2 & e^{-\hat{\varphi}} \cos \rho/2 \end{pmatrix}, \\
\text{Region III} \quad : \quad g &= e^{-i\theta_L \tau^2} s e^{-i\rho \tau^1} e^{-i\theta_R \tau^2} = \begin{pmatrix} e^{\hat{\varphi}} \sinh \rho/2 & e^{\hat{t}} \cosh \rho/2 \\ -e^{-\hat{t}} \cosh \rho/2 & -e^{-\hat{\varphi}} \sinh \rho/2 \end{pmatrix},
\end{aligned} \tag{2.18}$$

where  $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,

$$\theta_L = \hat{\varphi} + \hat{t}, \quad \theta_R = \hat{\varphi} - \hat{t}, \tag{2.19}$$

and

$$\begin{aligned}
\text{Region I} \quad : \quad \hat{r} &= \cosh \rho/2, & \sqrt{\hat{r}^2 - 1} &= \sinh \rho/2, & (\rho > 0), \\
\text{Region II} \quad : \quad \hat{r} &= \cos \rho/2, & \sqrt{1 - \hat{r}^2} &= \sin \rho/2, & (\pi > \rho > 0), \\
\text{Region III} \quad : \quad \sqrt{-\hat{r}^2} &= \sinh \rho/2, & \sqrt{1 - \hat{r}^2} &= \cosh \rho/2, & (\rho > 0).
\end{aligned} \tag{2.20}$$

The currents (2.10) then take the form, e.g.,

$$J^2 = \frac{k}{2} (\partial \theta_L + (2\hat{r}^2 - 1) \partial \theta_R), \quad \tilde{J}^2 = \frac{k}{2} (\bar{\partial} \theta_R + (2\hat{r}^2 - 1) \bar{\partial} \theta_L). \tag{2.21}$$

The translations of  $\hat{t}$  and  $\hat{\varphi}$  are generated by the linear combinations  $J_0^2 \pm \tilde{J}_0^2$  from (2.11). The translation of  $\varphi$  is generated by  $Q_\varphi \equiv \Delta_- J_0^2 + \Delta_+ \tilde{J}_0^2$ , where  $\Delta_\pm = r_+ \pm r_-$ . In terms of  $\theta_L$  and  $\theta_R$ ,  $\delta\varphi = 2\pi$  with fixed  $t$  is expressed by

$$\Delta_+ \delta \theta_L = \Delta_- \delta \theta_R = 2\pi \Delta_+ \Delta_- . \tag{2.22}$$

To describe the black hole, we have to twist (orbifold) the WZW model with respect to this discrete action. In the following, we will call our black hole the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  black hole.

### 3 The spectrum of the $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$ orbifold

As a consequence of the identification  $\varphi \sim \varphi + 2\pi$ , twisted (winding) sectors arise in this theory. In this section, we will discuss the spectrum including the twisted sectors. In orbifolding, the level matching is required from the consistency of string theory, for example, modular invariance and the invariance under the shift of the world-sheet spatial coordinate. In addition, we have to check the other consistency conditions such as unitarity. These consistency conditions are closely related to each other.

One difficulty to construct the orbifold is that the field  $\varphi$  is not a free field. We are working in a group manifold, so we cannot use the argument for flat theories. However, a similar orbifolding has been discussed in [23] to construct a  $SU(2)/\mathbf{Z}_N$  orbifold. We will follow their argument and solve the level matching condition explicitly. Since we are dealing with a non-compact group manifold, there are subtleties as a sensible string theory. We will return to these issues later.

### 3.1 Kac-Moody Primaries in the $\widetilde{SL}(2, \mathbf{R})$ WZW model

Before discussing the orbifold, let us consider Kac-Moody primaries in the  $\widetilde{SL}(2, \mathbf{R})$  WZW model. Operators are Kac-Moody primary if they form irreducible representations of global  $\widetilde{SL}(2, \mathbf{R})_L \times \widetilde{SL}(2, \mathbf{R})_R$  and if they are annihilated by the Kac-Moody generators  $J_n^a$  and  $\tilde{J}_n^a$  for  $n > 0$ . For WZW models, they are also Virasoro primary. For a compact group, local fields (wave functions) on the group correspond to Kac-Moody primaries [23, 24]. Thus, we make an ansatz [4, 6] that the Kac-Moody primary fields are given by local expressions in the fields  $\theta_L, \theta_R$  and  $\rho$ , but do not contain derivatives of these fields. Hence, they take the form

$$V(\theta_L(z, \bar{z}), \theta_R(z, \bar{z}), \rho(z, \bar{z})) . \quad (3.1)$$

Furthermore, we assume [4] that the Kac-Moody primary fields lead to normalizable operators, and that the CFT inherits the natural inner product of the  $\widetilde{SL}(2, \mathbf{R})$  representations. A complete basis for the square integrable functions on  $\widetilde{SL}(2, \mathbf{R})$  is known in the mathematical literature. It is given by the matrix elements of the following unitary representations; the principal continuous series, the highest and lowest weight discrete series [25, 4]. Thus, the objects satisfying our requirements are the matrix elements of the above unitary representations and they provide the primary fields in the  $\widetilde{SL}(2, \mathbf{R})$  CFT. We have summarized useful properties of  $\widetilde{SL}(2, \mathbf{R})$  representations in Appendix A.

Note that our choice of the primary fields corresponds to taking a unitary  $\widetilde{SL}(2, \mathbf{R})$  representation as a base of the Kac-Moody module. Most of our discussion below does not change even if we start with the other representations at the base including non-unitary ones as in [5].

In representations of  $\widetilde{SL}(2, \mathbf{R})$ , we have three types of basis. Let us denote the generators of  $sl(2, \mathbf{R})$  by  $J^0, J^1$  and  $J^2$ . The bases diagonalizing  $J^0, J^2$  and  $J^0 - J^1$  are called elliptic, hyperbolic and parabolic respectively. Since we are interested in the orbifolding related to the action of  $J_0^2$  and  $\tilde{J}_0^2$ , we consider representations in the hyperbolic basis. This basis has been used in the study of the Minkowskian  $SL(2, \mathbf{R})/U(1)$  black hole [4, 5]. We denote three types of primary fields, i.e., the matrix elements by

$$\begin{aligned} {}^P D_{J^\pm, J'^\pm}^\chi(g) & \quad \text{for the principal continuous series,} \\ {}^{H(L)} D_{J, J'}^j(g) & \quad \text{for the highest (H) and the lowest (L) series,} \end{aligned} \quad (3.2)$$



where  $j$  labels the value of the Casimir;  $J$  and  $J'$  refer to the eigenvalue of  $J^2$ . For the principal continuous series, we have additional parameters,  $0 \leq m_0 < 1$  specifying the representation, and  $\pm$  specifying the base state.  $\chi$  is the pair  $(j, m_0)$ . Under this construction, the primary fields have the common  $j$ -value in the left and right sector. Note that the spectrum of  $J^2$  ranges all over the real number, namely  $J, J' \in \mathbf{R}$ . For the details, see Appendix A.

### 3.2 Primary fields in the $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$ black hole CFT

We now turn to the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  CFT. The currents  $J^2(z)$  and  $\tilde{J}^2(\bar{z})$  are chiral and have the operator product expansions (OPE)

$$J^2(z)J^2(0) \sim \frac{k/2}{z^2}, \quad \tilde{J}^2(\bar{z})\tilde{J}^2(0) \sim \frac{k/2}{\bar{z}^2}. \quad (3.3)$$

So, we represent them by free fields  $\theta_L^F(z)$  and  $\theta_R^F(\bar{z})$  as

$$J^2(z) = \frac{k}{2} \partial \theta_L^F, \quad \tilde{J}^2(\bar{z}) = \frac{k}{2} \bar{\partial} \theta_R^F. \quad (3.4)$$

The normalization of the fields is fixed by

$$\theta_L^F(z)\theta_L^F(0) \sim +\frac{2}{k} \ln z, \quad \theta_R^F(\bar{z})\theta_R^F(0) \sim +\frac{2}{k} \ln \bar{z}. \quad (3.5)$$

The signs are opposite to the usual case due to the negative metric of the  $J^2$  direction. The explicit forms of  $\theta_L^F$  and  $\theta_R^F$  are obtained by integration of (2.21). The local integrability is assured by the current conservation. We also introduce  $\theta_L^{NF}(z, \bar{z})$  and  $\theta_R^{NF}(z, \bar{z})$  by

$$\theta_L(z, \bar{z}) = \theta_L^F(z) + \theta_L^{NF}(z, \bar{z}), \quad \theta_R(z, \bar{z}) = \theta_R^F(\bar{z}) + \theta_R^{NF}(z, \bar{z}). \quad (3.6)$$

Note  $\theta_L^{NF}$  and  $\theta_R^{NF}$  are not free fields.

Now, consider the operator

$$W_n(z, \bar{z}) \equiv \exp \left\{ -i \frac{k}{2} n \left( \Delta_- \theta_L^F - \Delta_+ \theta_R^F \right) \right\}, \quad (3.7)$$

where  $n \in \mathbf{Z}$ . They have the OPE's

$$\begin{aligned} \theta_L^F(z)W_n(0, \bar{z}) &\sim -in\Delta_- \ln z \cdot W_n(0, \bar{z}), \\ \theta_R^F(\bar{z})W_n(z, 0) &\sim +in\Delta_+ \ln \bar{z} \cdot W_n(z, 0). \end{aligned} \quad (3.8)$$

Thus,  $\theta_L^F$  and  $\theta_R^F$  shift by  $2\pi\Delta_-n$  and  $2\pi\Delta_+n$ , respectively, under the translation of the world-sheet coordinate  $\sigma \rightarrow \sigma + 2\pi$ , i.e.,  $z \rightarrow e^{2\pi i}z$  and  $\bar{z} \rightarrow e^{-2\pi i}\bar{z}$ . Hence,  $\delta\varphi = 2\pi n$  and  $\delta t = 0$  on  $W_n(z, \bar{z})$  under  $\delta\sigma = 2\pi$ . Thus,  $W_n(z, \bar{z})$  expresses the twisting with winding number  $n$ .

A general untwisted primary field takes the form (3.2). In our parametrization (2.18), it is given by

$$V_{J_L, J_R}^{j,0}(z, \bar{z}) = D_{J_L, J_R}^j(g'(\rho)) e^{-iJ_L\theta_L - iJ_R\theta_R}, \quad (3.9)$$

where we have omitted irrelevant indices of the matrix elements. The explicit form of  $g'(\rho)$  depends on which region we consider. Combining the untwisted primary field and the twisting operator, we obtain the general primary field in the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  black hole CFT:

$$\begin{aligned} V_{J_L, J_R}^{j,n}(z, \bar{z}) &= V_{J_L, J_R}^{j,0}(z, \bar{z}) W_n(z, \bar{z}), \\ &= D_{J_L, J_R}^j(g'(\rho)) \exp \left\{ -i \left( J'_L \theta_L^F + J_L \theta_L^{NF} + J'_R \theta_R^F + J_R \theta_R^{NF} \right) \right\}, \end{aligned} \quad (3.10)$$

where

$$J'_L = J_L + \frac{k}{2} \Delta_- n, \quad J'_R = J_R - \frac{k}{2} \Delta_+ n. \quad (3.11)$$

### 3.3 Level matching

In the previous subsection, we obtained primary fields. So, a general vertex operator has the form

$$J_N \cdot \tilde{J}_{\tilde{N}} \cdot V_{J_L, J_R}^{j,n}(z, \bar{z}), \quad (3.12)$$

where  $J_N$  and  $\tilde{J}_{\tilde{N}}$  stand for generic products of the Kac-Moody generators  $J_{-n}^a$  and  $\tilde{J}_{-n}^a$  respectively. The untwisted part depends on  $\theta_L^F$  and  $\theta_R^F$  as  $\exp(-i\omega_L\theta_L^F - i\omega_R\theta_R^F)$  and the full operator as  $\exp(-i\omega'_L\theta_L^F - i\omega'_R\theta_R^F)$ , where

$$\omega_L^{(\prime)} = J_L^{(\prime)} + i(N_+ - N_-), \quad \omega_R^{(\prime)} = J_R^{(\prime)} + i(\tilde{N}_+ - \tilde{N}_-). \quad (3.13)$$

$N_{\pm}$  and  $\tilde{N}_{\pm}$  are the number of  $J_{-n}^{\pm}$  and  $\tilde{J}_{-n}^{\pm}$  respectively. Notice that the commutation relation (2.16) implies that  $J_{-n}^{\pm}(\tilde{J}_{-n}^{\pm})$  shifts  $\omega_L(\omega_R)$  by  $\pm i$ . This is one feature of the representations in the hyperbolic basis.<sup>5</sup> If  $\omega_{L,R}^{(\prime)}$  are complex, the vertex operator cannot be single-valued on the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  manifold. Thus, we will focus on the vertex operators with  $N_+ = N_-$  and  $\tilde{N}_+ = \tilde{N}_-$ , namely,  $\omega_{L,R}^{(\prime)} = J_{L,R}^{(\prime)}$ .

The conformal dimension of the vertex operator is obtained by the GKO decomposition of the Virasoro algebra. Decompose the holomorphic part of the stress tensor as

$$\begin{aligned} T(z) &= T^{sl(2, \mathbf{R})/so(1,1)}(\rho, \theta_L^{NF}, \theta_R^{NF}) + T^{so(1,1)}(\theta_L^F), \\ T^{so(1,1)}(\theta_L^F) &= +\frac{k}{4} \partial \theta_L^F \partial \theta_L^F, \quad T^{sl(2, \mathbf{R})/so(1,1)} = T - T^{so(1,1)}. \end{aligned} \quad (3.14)$$

---

<sup>5</sup> This seems to be contradict the Hermiticity of  $J_0^2(\tilde{J}_0^2)$ . However, this is not the case because the spectrum of  $J_0^2(\tilde{J}_0^2)$  is continuous. Representations of  $SL(2, \mathbf{R})$  in the hyperbolic basis has been described in Appendix A.

Since  $T^{so(1,1)}$  acts only on  $\theta_L^F$ , the weight with respect to  $T^{so(1,1)}$  is given by  $\Delta^{so(1,1)}(J'_L) \equiv -J'^2_L/k +$  (the grade of  $J_{-n}^2$ 's). So, we have

$$\begin{aligned} L_0 &= \Delta^{sl(2,\mathbf{R})/so(1,1)}(j, J_L) + \Delta^{so(1,1)}(J'_L) \\ &= \frac{-j(j+1)}{k-2} + \frac{J_L^2 - J'^2_L}{k} + N, \end{aligned} \quad (3.15)$$

where  $-j(j+1)$  is the Casimir,  $N$  is the total grade of  $J_{-n}^a$ 's and  $\Delta^{sl(2,\mathbf{R})/so(1,1)}$  is the weight with respect to  $T^{sl(2,\mathbf{R})/so(1,1)}$ . Here, we have used that  $L_0$  is given by the Casimir plus the total grade for the untwisted sector, namely,

$$\Delta^{sl(2,\mathbf{R})/so(1,1)}(j, J_L) + \Delta^{so(1,1)}(J_L) = -\frac{j(j+1)}{k-2} + N. \quad (3.16)$$

Similarly, we obtain

$$\tilde{L}_0 = \frac{-j(j+1)}{k-2} + \frac{J_R^2 - J'^2_R}{k} + \tilde{N}. \quad (3.17)$$

We are now ready to solve the level matching condition. The condition is

$$L_0 - \tilde{L}_0 = -n \left[ (\Delta_- J_L + \Delta_+ J_R) - \frac{k}{2} n J_{BH} \right] + N - \tilde{N} \in \mathbf{Z}. \quad (3.18)$$

Furthermore, consider the OPE of two vertex operators with quantum numbers  $(n_i, J_{L,i}, J_{R,i})$  ( $i = 1, 2$ ). Since  $J_{L,R}$  and  $n$  are conserved, the level matching condition for the resulting operator reads

$$-(n_1 + n_2) \sum_{i=1}^2 \left[ (\Delta_- J_{L,i} + \Delta_+ J_{R,i}) - \frac{k}{2} n_i J_{BH} \right] \in \mathbf{Z}. \quad (3.19)$$

Therefore, if  $J_{L(R),1(2)}$  and  $n_{1,2}$  satisfy (3.18), the closure of the OPE requires

$$(\Delta_- J_L + \Delta_+ J_R) - \frac{k}{2} n J_{BH} \equiv m \in \mathbf{Z}. \quad (3.20)$$

This is the solution to the level matching condition.

We can check single-valuedness of the vertex operator which satisfies this condition. Let us denote the  $\theta_{L,R}^{F,NF}$ -dependence of (3.10) by  $\exp(-i\Theta)$  and recall (2.22). Then, under  $\delta\varphi = 2\pi$ ,

$$\delta\Theta = 2\pi m + \frac{k}{2}\pi n \left[ \frac{1}{\pi} \left( \Delta_- \delta\theta_L^F - \Delta_+ \delta\theta_R^F \right) + \left( \Delta_+^2 - \Delta_-^2 \right) \right]. \quad (3.21)$$

Hence, the vertex operator is invariant under

$$\delta\theta_L^{NF} = \delta\theta_L^F = \pi\Delta_-, \quad \delta\theta_R^{NF} = \delta\theta_R^F = \pi\Delta_+. \quad (3.22)$$

Single-valuedness is guaranteed in this sense.

In our twisting, only the free field part seems relevant. In the untwisted sector, only the combinations  $\theta_{L,R} = \theta_{L,R}^F + \theta_{L,R}^{NF}$  appear, so this does not matter. On the other hand, for a twisted sector, this is curious because we were originally considering the orbifolding with respect to  $\varphi \sim \varphi + 2\pi$  including the non-free part. However, the non-free part *is* relevant in the above sense. This is related to the Noether current ambiguity in field theory [23]. In any case, one can take the point of view that we are just considering possible degrees of freedom represented by the twisting with respect to  $\theta_{L,R}^F$ .

So far we have dealt with a generic value of  $\Delta_{\pm}$  corresponding to a rotating black hole. For the non-rotating black hole, set  $\Delta_+ = \Delta_- = r_+$  in the above discussion. Also, we can formally take the extremal limit  $\Delta_- \rightarrow 0$  at the end. However, the procedure to get the extremal black hole from  $\widetilde{SL}(2, \mathbf{R})$  is different from the non-extremal one [18] and singular quantities appear in the course of the discussion. Thus we have to examine whether the extremal limit in our result correctly represents the extremal limit.

### 3.4 Physical states

Let us turn to discussion on physical states. We use the old covariant quantization. The base states corresponding to the primary fields are written as

$$|j; J_L, n\rangle |j; J_R, n\rangle. \quad (3.23)$$

The Kac-Moody module is built on this base by  $J_{-n}^a$ . Since the eigenvalues of  $J_0^2$  and  $\tilde{J}_0^2$  should be real in our case, the number of  $J_{-n}^{\pm}$  is restricted as mentioned above. Then the Kac-Moody module of the holomorphic part is spanned by states

$$K_{-1}^a K_{-1}^b \cdots K_{-1}^c |j; J_L, n\rangle, \quad (3.24)$$

where  $K_{-1}^a$  ( $a = +, -, 2$ ) are defined by

$$K_{-1}^+ = J_{-1}^+ J_0^-, \quad K_{-1}^- = J_{-1}^- J_0^+, \quad K_{-1}^2 = J_{-1}^2. \quad (3.25)$$

The states obtained by acting  $J_0^a$  on the above states are excluded unless they result in the above form.

The physical states consist of the left and the right part of the form (3.24) and satisfy the physical state conditions

$$(L_n - \delta_n) |\Psi\rangle = (\tilde{L}_n - \delta_n) |\Psi\rangle = 0 \quad (n \geq 0). \quad (3.26)$$

The on-shell condition yields

$$J_L = -\frac{k}{4}\Delta_- n + \frac{1}{\Delta_- n} \left( N - 1 - \frac{j(j+1)}{k-2} \right),$$

$$\begin{aligned}
J_R &= +\frac{k}{4}\Delta_{+n} - \frac{1}{\Delta_{+n}} \left( \tilde{N} - 1 - \frac{j(j+1)}{k-2} \right), \\
N &= \tilde{N} + nm
\end{aligned}
\tag{3.27}$$

for twisted sectors  $n \neq 0$ , and

$$1 = -\frac{j(j+1)}{k-2} + N, \quad N = \tilde{N} \tag{3.28}$$

for the untwisted sector. Therefore, for a given  $j$ , an arbitrarily excited state is allowed in twisted sectors. On the contrary, in the untwisted sector,  $j$ -value is completely determined by grade  $N$ :

$$j = j(N) \equiv \frac{1}{2} \left\{ -1 - \sqrt{1 + 4(k-2)(N-1)} \right\}, \tag{3.29}$$

where we have chosen the branch  $\text{Re } j \leq -1/2$  (see Appendix A). This result is the same as in the string theory on  $SL(2, \mathbf{R})$ .

## 4 Investigation of unitarity

In the previous section, we discussed the spectrum of the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  orbifold by solving the level matching condition. But there are other consistency conditions we must take into account, and as a result, the spectrum in Sec. 3 may be further restricted.

In this section, we will investigate the ghost problem. The unitary (ghost) problem for the string on  $SL(2, \mathbf{R})$  has been discussed and it is shown to contain ghosts [9],[15, 16]. There are attempts to get a unitary theory by restricting the spectrum [11]. Also, a unitary  $SL(2, \mathbf{R})$  theory has been proposed using modified currents [16].

In our case, the analysis of unitarity is different from the string theory on  $SL(2, \mathbf{R})$  due to the existence of winding modes and the use of representations in the hyperbolic basis. However, we can still utilize a tool developed for the  $SL(2, \mathbf{R})$  theory with a slight modification. Thus, we will first summarize the argument for the  $SL(2, \mathbf{R})$  case. Then, we will show the non-unitarity of the string on  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  orbifold by constructing physical states with negative norms.

### 4.1 The unitarity problem of the string on $SL(2, \mathbf{R})$

Let us briefly review the unitarity problem in the  $SL(2, \mathbf{R})$  case [9],[15, 16]. The holomorphic and the anti-holomorphic parts are independent in the  $SL(2, \mathbf{R})$  WZW model until we consider the modular properties, so we focus on the holomorphic part. In order to study the unitarity problem of the  $SL(2, \mathbf{R})$  theory, it is useful to notice the following facts:

1. The on-shell condition is the same as (3.28) or (3.29).
2. Let  $V^a$  be an operator satisfying

$$[I_0^a, V^b] = i\epsilon^{ab}_c V^c. \quad (4.1)$$

(An example is  $V^a = I_{-n}^a$ .) Consider the following states:

$$V^+ I_0^- |j; m\rangle, \quad V^- I_0^+ |j; m\rangle, \quad V^0 |j; m\rangle, \quad (4.2)$$

where  $|j; m\rangle$  is an eigenstate with the Casimir  $\mathbf{C} = -j(j+1)$  and  $I_0^0 = m$  (not necessarily base states). Assume they do not vanish. Then, by evaluating the matrix elements of the Casimir operator, we find that these states are decomposed into the representations of  $sl(2, \mathbf{R})$  with the  $j$ -values  $j$  and  $j \pm 1$ .

3. As a consequence of 2), acting  $I_{-1}^a$   $N$  times on a base state  $|j; m\rangle$  yields  $3^N$  independent states at grade  $N$  with  $j$ -values ranging from  $j - N$  to  $j + N$ . Let us call the states with  $j \pm N$  the “extremal states” and denote them by  $|E_N^\pm\rangle$ .  $|E_N^\pm\rangle$  are physical if they satisfy the on-shell condition. The reason is simple. Since the Casimir commutes with  $L_n$ ,  $L_n |E_N^\pm\rangle$  have the same  $j$ -values as  $|E_N^\pm\rangle$ . However,  $L_n |E_N^\pm\rangle$  are at grade  $N - n$ , and thus their  $j$ -values should range from  $j - (N - n)$  to  $j + (N - n)$ . Therefore, we have  $L_n |E_N^\pm\rangle = 0$  ( $n > 0$ ); together with the on-shell condition, they are physical.

4. Let  $|\Psi\rangle$  be a physical state, i.e.,  $(L_n - \delta_n) |\Psi\rangle = 0$  for  $n \geq 0$ . Then the states obtained by acting  $J_0^a$  on  $|\Psi\rangle$  are also physical:

$$L_n J_0^a \cdots J_0^b |\Psi\rangle = [L_n, J_0^a \cdots J_0^b] |\Psi\rangle = 0. \quad (4.3)$$

5. For the discrete series, we have a simple expression of the extremal states, e.g.,

$$|E_N^{d+}\rangle = (I_{-1}^+)^N |j(N); j(N)\rangle, \quad (4.4)$$

where  $|j(N); j(N)\rangle$  is a highest-weight state, namely  $I_0^+ |j(N); j(N)\rangle = 0$ . It is easy to obtain the norm of this state:

$$\langle E_N^{d+} | E_N^{d+} \rangle = \langle j(N); j(N) | j(N); j(N) \rangle (N!) \prod_{r=0}^{N-1} (k + 2j(N) + r). \quad (4.5)$$

We can immediately find physical states with negative norms. First, let us consider the case  $k < 2$ . From 1) and 3),  $|E_N^+\rangle$  with  $j = j(N)$  at its base is a physical state. At sufficiently large  $N$ ,  $j(N)$  takes a value of the principal continuous series. On the other hand, the  $j$ -value of  $|E_N^+\rangle$  is  $j(N) + N$ . There is no unitary representations with this  $j$ -value. Thus, the module  $I_0^a \cdots I_0^b |E_N^+\rangle$  is physical, but forms a *non-unitary* representation of  $sl(2, \mathbf{R})$ .

Second, let us consider the case  $k > 2$ . Again  $|E_N^{d+}\rangle$  with  $j = j(N)$  at its base is a physical state. We easily find that

$$I_0^+ |E_N^{d+}\rangle = 0, \quad I_0^0 |E_N^{d+}\rangle = (j(N) + N) |E_N^{d+}\rangle. \quad (4.6)$$

Thus  $|E_N^{d+}\rangle$  is a highest-weight state of a highest-weight  $sl(2, \mathbf{R})$  representation like  $|j(N) + N; j(N) + N\rangle$ . However, the  $I^0$ -value becomes positive for large  $N$ . Since there is no unitary representation of  $sl(2, \mathbf{R})$  with such a highest weight state, the states in the  $sl(2, \mathbf{R})$  representation built on  $|E_N^+\rangle$  by  $I_0^a$  are physical but some have negative norms.

Although we can flip the sign of the norm of  $|j(N); j(N)\rangle$  so that  $\langle E_N^{d+} | E_N^{d+} \rangle > 0$  for arbitrary  $N$ , we cannot remove physical states with negative norms. This is because we have infinitely many physical states built on  $|E_N^{d+}\rangle$  as in 4), and they form a non-unitary  $sl(2, \mathbf{R})$  representation.

## 4.2 Physical states up to grade 1

Now let us discuss the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  orbifold case. One difference from the previous discussion is the existence of winding modes. Thus, for twisted sectors, (3.29) does not hold and the holomorphic and anti-holomorphic part are not independent. The other important difference is that we use the hyperbolic basis and the Kac-Moody module is restricted to the form (3.24). We do not have states given in 4) and 5) in the previous subsection. Nevertheless, the argument on extremal states is still valid, so we will use them to show that our theory is not unitary.

First, let us consider physical states up to grade one. For the time being, we focus on the holomorphic part. At grade one, we have three states for a fixed  $j$ -,  $J_0^2$ - and  $n$ - value, namely,

$$|\pm\rangle \equiv K_{-1}^{\pm} |j; \lambda, n\rangle, \quad |2\rangle \equiv K_{-1}^2 |j; \lambda, n\rangle. \quad (4.7)$$

Using the Hermiticity (2.17) and (A.22), we get norms among the above states

$$\begin{pmatrix} \langle + | \\ \langle - | \\ \langle 2 | \end{pmatrix} (|+\rangle, |-\rangle, |2\rangle) = \begin{pmatrix} 0 & -(2i\lambda + k')A & -iA \\ (2i\lambda - k')\bar{A} & 0 & i\bar{A} \\ i\bar{A} & -iA & k/2 \end{pmatrix}, \quad (4.8)$$

where  $A = j(j+1) + \lambda(\lambda+i)$  and  $k' = k - 2$ . We have omitted  $\langle j; \lambda, n | j; \lambda, n \rangle$ . These states are decomposed into the eigenstates of the Casimir with  $j$ -values  $j$  and  $j \pm 1$  from the argument in Appendix B. We denote them by  $|\Phi^j(j; \lambda, n)\rangle$  and  $|\Phi^{j\pm 1}(j; \lambda, n)\rangle$ . Note  $|\Phi^{j\pm 1}\rangle$  are extremal states. Explicitly, they are given by (up to normalization)

$$\begin{pmatrix} |\Phi^{j+1}\rangle \\ |\Phi^j\rangle \\ |\Phi^{j-1}\rangle \end{pmatrix} = \begin{pmatrix} j+1-i\lambda & -(j+1+i\lambda) & 2i((j+1)^2 + \lambda^2) \\ 1 & 1 & -2\lambda \\ -(j+i\lambda) & j-i\lambda & 2i(j^2 + \lambda^2) \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \\ |2\rangle \end{pmatrix}. \quad (4.9)$$

At grade one, the conditions  $L_n = 0$  ( $n > 0$ ) reduce to  $L_1 = 0$ . This imposes one equation on a state given by a linear combination of  $|\pm\rangle$  and  $|2\rangle$ . Then the space of the solution has (complex) two dimensions at a generic value of  $j$  and  $\lambda$ . Since extremal states are physical and we have two extremal states at grade one, the physical states take the form

$$\alpha |\Phi^{j+1}\rangle + \beta |\Phi^{j-1}\rangle. \quad (4.10)$$

At special values of  $\lambda$  and  $j$ , we have extra solutions. Similarly, we can get the states at grade one satisfying the  $\tilde{L}_1 = 0$  condition. Hence, the physical states up to grade one are obtained by tensoring the holomorphic and the anti-holomorphic part using the states (4.10) and base states which satisfy the on-shell condition (3.27) or (3.28).

### 4.3 Non-unitarity of the string on $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$ orbifold

Finding physical states with negative norms is easy using the above physical states. First, let us discuss the case of real  $j$  (the discrete series).<sup>6</sup> There exist the following physical states:

$$|\Psi_1^d\rangle = |j; J_{L,1}, 1\rangle |j; J_{R,1}, 1\rangle, \quad |\Psi_2^d\rangle = |\Phi^{j+1}(j; J_{L,2}, 1)\rangle |j; J_{R,2}, 1\rangle \quad (4.11)$$

where  $m_1 = 0$ ,  $m_2 = 1$  and

$$\begin{aligned} J_{L,1} &= -\frac{k}{4}\Delta_- - \frac{1}{\Delta_-} \left(1 + \frac{j(j+1)}{k-2}\right), & J_{R,1} &= \frac{k}{4}\Delta_+ + \frac{1}{\Delta_+} \left(1 + \frac{j(j+1)}{k-2}\right), \\ J_{L,2} &= -\frac{k}{4}\Delta_- - \frac{1}{\Delta_-} \frac{j(j+1)}{k-2}, & J_{R,2} &= J_{R,1}. \end{aligned} \quad (4.12)$$

By explicit calculation, norms of these states are

$$\begin{aligned} \langle \Psi_1^d | \Psi_1^d \rangle &= \langle j; J_{L,1}, 1 | j; J_{L,1}, 1 \rangle \langle j; J_{R,1}, 1 | j; J_{R,1}, 1 \rangle, \\ \langle \Psi_2^d | \Psi_2^d \rangle &= 2(j+1)(2j+1)(2j+k) \left( (j+1)^2 + J_{L,2}^2 \right) \\ &\quad \times \langle j; J_{L,2}, 1 | j; J_{L,2}, 1 \rangle \langle j; J_{R,2}, 1 | j; J_{R,2}, 1 \rangle. \end{aligned} \quad (4.13)$$

If the bases of  $|\Psi_1^d\rangle$  and  $|\Psi_2^d\rangle$  belong to the same representation of  $sl(2, \mathbf{R})$ ,  $\langle j; J_{L,i}, 1 | j; J_{L,i}, 1 \rangle \langle j; J_{R,i}, 1 | j; J_{R,i}, 1 \rangle$  ( $i = 1, 2$ ) take the same value. Then, for a sufficiently large  $|j|$  (recall  $j \leq -1/2$ ), the latter norm behaves as  $8j^7/(k'\Delta_-)^2$ , and the two norms have opposite signs. Thus, if we include the bases with real  $j$ , our orbifold cannot be unitary.

Next, let us turn to the case of complex  $j$  (the principle continuous series). Because  $j = -1/2 + i\nu$  ( $\nu > 0$ ), the extremal states at grade one have  $j = -1/2 \pm 1 + i\nu$ . These

---

<sup>6</sup>This argument is also valid for the complementary series although we have not included this series.



correspond to complex Casimirs and non-unitary  $sl(2, \mathbf{R})$  representations. This is not the end of the story however because (i) infinite series of states build on these states by the current zero-modes are not allowed and (ii) the left and right sector are connected by the quantum numbers  $n$  and  $m$ . In this case, the norm of  $|\Psi_2^d\rangle$  vanishes. (Thus we have infinitely many physical states with zero norm; see Appendix B.) Hence, consider the following physical states instead:

$$\begin{aligned} |\Psi_1^p\rangle &= |j; J_L^1, 1\rangle |j; J_R^1, 1\rangle, \\ |\Psi_2^p\rangle &= \left( |\Phi^{j-1}(j; J_{L,2}, 1)\rangle - i |\Phi^{j+1}(j; J_{L,2}, 1)\rangle \right) |j; J_{R,2}, 1\rangle, \end{aligned} \quad (4.14)$$

where  $J_{L(R),i}$  are given by (4.12). The norms of these states are

$$\begin{aligned} \langle \Psi_1^p | \Psi_1^p \rangle &= \langle j; J_{L,1}, 1 | j; J_{L,1}, 1 \rangle \langle j; J_{R,1}, 1 | j; J_{R,1}, 1 \rangle, \\ \langle \Psi_2^p | \Psi_2^p \rangle &= -4\nu \left[ \left( J_{L,2}^2 - 1/4 - \nu^2 \right) \left( 4\nu^2 - 3k - 1 \right) + 2(1+k)J_{L,2}^2 - k \right] \\ &\quad \times \langle j; J_{L,2}, 1 | j; J_{L,2}, 1 \rangle \langle j; J_{R,2}, 1 | j; J_{R,2}, 1 \rangle. \end{aligned} \quad (4.15)$$

For a sufficiently large  $\nu$ , the latter norm behaves as  $-16\nu^7/(k'\Delta_-)^2$ . Thus, the two norms have opposite signs if the bases of  $|\Psi_1^p\rangle$  and  $|\Psi_2^p\rangle$  belong to the same representation of  $sl(2, \mathbf{R})$ . Therefore if we include the bases belonging to the principal continuous series, our orbifold is again non-unitary.

Notice that bases with large  $|j|$  or  $\nu$  are generated from those of small values by tensor products (see Appendix A) unless they decouple.

For the  $SL(2, \mathbf{R})$  theory, a physical state at a sufficiently high grade has large  $|j|$  at the base, which caused the trouble. In our case, some ghosts in the  $SL(2, \mathbf{R})$  theory disappear, but physical states with large  $|j|$  at the base exist already at grade one due to the winding modes. We have still possibilities that the orbifold becomes ghost-free, for instance, by some truncation of the spectrum. We will discuss this issue in Sec. 6.

## 5 Tachyon and target-space geometry

In this section, we discuss the tachyon propagation on the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  black hole and the target-space geometry, which are irrelevant to the details of the full spectrum.

From the group theory point of view, the  $SL(2, \mathbf{R})/U(1)$  black hole and the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  black hole are closely related. For example, primary fields in both theories are constructed from the matrix elements of  $SL(2, \mathbf{R})$ . Hence we observe similar properties for the tachyon and the target-space geometry in two theories.

### 5.1 Tachyon in the untwisted sector

First, let us consider the tachyon in the untwisted sector. It is expressed by the matrix elements of  $\widetilde{SL}(2, \mathbf{R})$  in various unitary representations as (3.9). The matrix elements

satisfy the differential equation [25]

$$[\Delta - j(j+1)] D_{J_L, J_R}^{j(\chi)}(g) = 0, \quad (5.1)$$

where  $\Delta$  is the Laplace operator on  $SL(2, \mathbf{R})$ . Because the geometry of the black hole is locally  $SL(2, \mathbf{R})$ , this equation is nothing but the linearized tachyon equation [26] or the Klein-Gordon equation in the BTZ black hole background [27] up to a factor. Then the analysis of the tachyon scattering and the Hawking radiation in [26] is valid without change. We do not repeat it here, but only make an explicit correspondence between the untwisted tachyon in [26] and ours.  $j(j+1)$  represents the mass-squared. For the untwisted sector, the on-shell condition is (3.28) with  $N = \tilde{N} = 0$ , so gives a principal continuous series.

In [26], the tachyon is expanded as

$$T = \sum_{N \in \mathbf{Z}} \int dE T_{EN}(r) e^{-iEt} e^{-iN\varphi} = \sum_{\hat{E}, \hat{N}} T_{\hat{E}\hat{N}}(\hat{r}) e^{-i\hat{E}t} e^{-i\hat{N}\hat{\varphi}}. \quad (5.2)$$

After changes of variables to  $z = 1 - \hat{r}^2$  and

$$T_{\hat{E}\hat{N}}(z) = z^{i\hat{E}/2} (1-z)^{i\hat{N}/2} \Psi_{\hat{E}\hat{N}}(z), \quad (5.3)$$

we find that  $\Psi_{\hat{E}\hat{N}}(z)$  is given by the hypergeometric function. Comparing the above expression with (3.9), we get the correspondences

$$\begin{aligned} T &\leftrightarrow {}^P D_{J_L \pm, J_R \pm}^{\chi}(g), & T_{\hat{E}\hat{N}} &\leftrightarrow {}^P D_{J_L \pm, J_R \pm}^{\chi}(g'), \\ J_L + J_R &\leftrightarrow \hat{N}, & J_L - J_R &\leftrightarrow \hat{E}, \end{aligned} \quad (5.4)$$

where we have used (2.19). Since  $\varphi$  has period  $2\pi$ ,  $N = -r_- \hat{E} + r_+ \hat{N} \in \mathbf{Z}$ . This is the level matching condition (3.20) with  $n = 0$ .

As a further check, let us consider the matrix elements for  $g' = \begin{pmatrix} \cosh \rho/2 & \sinh \rho/2 \\ \sinh \rho/2 & \cosh \rho/2 \end{pmatrix}$  ( $\rho > 0$ ); this corresponds to the region  $r > r_+$ . They are given by

$$\begin{aligned} {}^P D_{J_L+, J_R+}^{\chi}(g') &= \frac{1}{2\pi} B(\mu_L, -\mu_L - 2j) \frac{\cosh^{2j+\mu_L+\mu_R} \rho/2}{\sinh^{\mu_L+\mu_R} \rho/2} F(\mu_L, \mu_R; -2j; -\sinh^{-2} \rho/2), \\ {}^P D_{J_L-, J_R-}^{\chi}(g') &= \frac{1}{2\pi} B(1 - \mu_R, \mu_R - 1 + 2(j+1)) \frac{\cosh^{2j+\mu_L+\mu_R} \rho/2}{\sinh^{4j+2+\mu_L+\mu_R} \rho/2} \\ &\quad \times F(\mu_L + 2j + 1, \mu_R + 2j + 1; 2j + 2; -\sinh^{-2} \rho/2), \end{aligned} \quad (5.5)$$

where  $\mu_{L,R} = iJ_{L,R} - j$ .  $F$  and  $B$  are the hypergeometric function and the Euler beta function respectively. Noting  $-\sinh^2 \rho/2 = 1 - \hat{r}^2 = z$ , we find that  ${}^P D_{J_L+, J_R+}^{\chi}(g')$  and  ${}^P D_{J_L-, J_R-}^{\chi}(g')$  are the mode functions in [26] which are regular at infinity. Generically,

the untwisted tachyon behaves as

$$\begin{aligned}
{}^P D_{J_L+, J_R+}^\chi(g') &\sim a_1 (r^2)^j && \text{as } r \rightarrow \infty, \\
&\sim b_1 e^{i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} + b_2 e^{-i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} && \text{as } r \rightarrow r_+, \\
{}^P D_{J_L-, J_R-}^\chi(g') &\sim a'_1 (r^2)^{-(j+1)} && \text{as } r \rightarrow \infty, \\
&\sim b'_1 e^{i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} + b'_2 e^{-i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} && \text{as } r \rightarrow r_+,
\end{aligned} \tag{5.6}$$

where  $a_1^{(')}$  and  $b_{1,2}^{(')}$  are some constants. Since  $\text{Re } j = -1/2$ , they behave like spherical waves asymptotically. When  $J_L = J_R$ , hypergeometric functions degenerate; then the asymptotic behaviors as  $r \rightarrow r_+$  are different from (5.6).

## 5.2 Tachyon in twisted sectors

Now let us turn to the tachyon in twisted sectors. The twisted tachyon is given by the product of the matrix element and the twisting operator as (3.10). The twisting operator gives a phase to the tachyon. In the twisted sectors, various  $j$ -values are allowed from the on-shell condition (3.27) with  $m = N = \tilde{N} = 0$  and  $n \neq 0$ . Thus the matrix elements of the discrete series appear as well as those of the principal continuous series. For the principal continuous series, the explicit forms and asymptotic behaviors of the matrix elements are the same as in the untwisted sector (although  $j$ -values are different).

For the discrete series, only one linear combination of solutions to (5.1) appears. As explained in Appendix A, the matrix elements are obtained from one of the matrix elements in the principal continuous series;

$${}^L D_{J_L, J_R}^j(g') \propto {}^H D_{J_L, J_R}^j(g') \propto {}^P D_{J_L+, J_R+}^\chi(g'). \tag{5.7}$$

Thus we can read off the behaviors of  ${}^{L,H} D_{J_L, J_R}^j(g')$  from  ${}^P D_{J_L+, J_R+}^\chi(g')$ . Note in particular that  ${}^{L,H} D_{J_L, J_R}^j(g') \rightarrow (r^2)^j$  and  $j \leq -1/2$ . Therefore, a tachyon state in the discrete series dumps rapidly as one goes to infinity, so this is a state localized near the black hole. This is similar to a winding state in the Euclidean  $SL(2, \mathbf{R})/U(1)$  black hole where one can regard it as a bound state in the dual geometry [4]. Hence, we have two kinds of tachyon. One is from the principal continuous series and propagates like a wave, and the other is from the discrete series and is localized near the black hole.

The differential equation (5.1) for the discrete series is again the Klein-Gordon equation. The thermodynamic properties of the corresponding scalar fields are discussed in [27].

## 5.3 Global properties

So far we have not discussed global properties of the tachyon, but considered the tachyon propagation in one patch of the orbifold (the region  $r > r_+$ ). In order to define the

tachyon propagation globally, we have to continue it from one region to another. Let us start with a tachyon in the region  $r > r_+$ . Recall that the regions have the boundaries at the inner and the outer horizon ( $r = r_{\pm}$ ). The tachyon is given by a linear combination of (5.5) or (5.7) and is regular at infinity. From the linear transformation formulas of hypergeometric functions, we can obtain the expression around  $r = r_{\pm}$  as in (5.6). We would like to continue it to the other regions.

Here we have two possible sources of obstacles. One is complex power of  $z$  or  $1 - z$ . This causes troubles as  $z \rightarrow 0$  ( $r \rightarrow r_+$ ) or  $z \rightarrow 1$  ( $r \rightarrow r_-$ ). The other is logarithmic singularities like  $\log z$  and  $\log(1 - z)$ . The logarithmic singularity at  $z = 0$  ( $r = r_+$ ) arises when  $\mu_L - \mu_R \in \mathbf{Z}$ , i.e.,  $J_L - J_R = 0$ , and the one at  $z = 1$  ( $r = r_-$ ) arises when  $\mu_L + \mu_R + 2j \in \mathbf{Z}$ , i.e.,  $J_L + J_R = 0$ . The latter corresponds to the case of the  $SL(2, \mathbf{R})/U(1)$  black hole in which the tachyon develops a logarithmic singularity at the origin (singularity) [4]. This is natural because the inner horizon of the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  black hole and the origin of the  $SL(2, \mathbf{R})/U(1)$  black hole are the same point in the  $SL(2, \mathbf{R})$  group manifold.

Note that the matrix elements are continuous all over the group manifold. Thus if we consider a generalized function space including distributions, we can continue the tachyon from one region to another in any case. We leave as open problems precise prescription of the continuation and the physical interpretation of the above singularities.

## 5.4 T-duality

The  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  black hole has two Killing vectors  $\partial_{\hat{t}}$  and  $\partial_{\hat{\varphi}}$ . In coordinates  $(\hat{t}, \hat{\varphi}, \hat{r})$ , the geometry is given by (2.7) and the dilaton  $\phi = 0$ . In order to deal with a general T-duality transformation, let us define new coordinates  $x$  and  $y$  by

$$\begin{pmatrix} \hat{t} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \alpha\delta - \beta\gamma \neq 0. \quad (5.8)$$

Then, the T-duality transformation to  $x$  covers all T-duality transformations.

First, let us consider the T-duality transformation to  $\varphi$ . This is discussed in [19]. Setting  $x = \varphi$  and  $y = t$ , the dual of the  $\widetilde{SL}(2, \mathbf{R})/\mathbf{Z}$  black hole becomes in general the black string. The T-duality transformation is not self-dual.

Next, let us set  $x = \hat{\varphi}$  and  $y = \hat{t} - \hat{\varphi}$ . In these coordinates, the geometry is given by

$$\begin{aligned} ds^2 &= \alpha'k \left\{ dx^2 + (1 - \hat{r}^2)dy^2 + 2(1 - \hat{r}^2)dxdy + (\hat{r}^2 - 1)^{-1}d\hat{r}^2 \right\}, \\ B &= \alpha'k \hat{r}^2 dx \wedge dy, \quad \phi = 0. \end{aligned} \quad (5.9)$$

The T-duality transformation [28, 29] gives the following dual geometry:

$$\begin{aligned} \tilde{ds}^2 &= \alpha'k \left\{ dx^2 + \hat{r}^2 dy^2 + 2\hat{r}^2 dxdy + (\hat{r}^2 - 1)^{-1}d\hat{r}^2 \right\}, \\ \tilde{B} &= \alpha'k(1 - \hat{r}^2)dx \wedge dy, \quad \tilde{\phi} = 0. \end{aligned} \quad (5.10)$$

This geometry is obtained from the original one via  $\hat{r}^2 \rightarrow 1 - \hat{r}^2$ , or  $\hat{t} \leftrightarrow \hat{\varphi}$ . Thus, this T-duality transformation is self-dual and interchanges the outside of the inner horizon<sup>7</sup> ( $\hat{r}^2 > 0$ ) and the inside of the outer horizon ( $\hat{r}^2 < 1$ ). In particular, the outer and the inner horizon (or the origin) are interchanged. Recall that translations of  $\hat{t}$  and  $\hat{\varphi}$  are the vector and the axial symmetry. So, the transformation  $\hat{t} \leftrightarrow \hat{\varphi}$  corresponds to the T-duality transformation in the  $SL(2, \mathbf{R})/U(1)$  black hole which interchanges the horizon and the singularity [4, 3].

Since  $\varphi$  is periodic, we have to further specify the periodicity of the dual coordinate. In the above T-duality transformation, the period of  $x = \hat{\varphi}$  in the dual geometry should be reciprocal of that in the original geometry [29]. From (2.8), we see that the periods of  $\hat{t}$  and  $\hat{\varphi}$  are not independent, so generically, we cannot specify the period of  $\hat{\varphi}$  only. However, for the non-rotating black hole ( $r_- = 0$ ), we have  $\hat{\varphi} = r_+ \varphi$  and the period of  $\hat{\varphi}$  in the original geometry is equal to  $2\pi r_+$ . Hence the period in the dual geometry is  $2\pi/(r_+ k)$ . This indicates that the black hole mass is reversed under the T-duality transformation because  $M_{BH} = r_+^2$ . Because  $J_{L,R}$  take all real values, the spectrum of  $L_0$  and  $\tilde{L}_0$  is formally invariant under this T-duality transformation. But it is not bounded from below as in Minkowski spacetime, so we need some procedure such as the Wick rotation for a rigorous argument.

## 6 Discussion

### 6.1 Consistency conditions

In Sec. 4, we found that there are physical states with negative norms. We can speculate various reasons why ghosts survive in our analysis:

1. Further truncation might be necessary on the spectrum.
2. Modular invariance might fix the problem.
3. The theory on  $SL(2, \mathbf{R})$  might be sick. The  $SL(2, \mathbf{R})$  WZW model describes anti-de Sitter space, so has unusual asymptotic properties.
4. One might have to use modified currents.
5. We might have to include non-unitary representations for base representations of current algebras.

All of possibilities listed above appear in the literature [5, 11, 12, 16]. However, the possibility 5) does not work: even if we include non-unitary representations, our argument

---

<sup>7</sup>or the outside of the origin for the non-rotating black hole

in Sec. 4 does not change very much and we can easily find physical states with negative norms. We will discuss the possibility 1) in the next subsection, which is different from previously discussed ones. But in this section, we first make comments on the other consistency conditions after making some general remark.

The basic physical consistency conditions for a string theory are not many. In general, as a sensible physical theory, we must require Lorentz invariance, a positive inner product for the observable Hilbert space and the unitary transition amplitude. There are few in number, but these in turn imply various consistency conditions such as world-sheet diffeomorphism and Weyl invariance, the absence of negative norm states, unitarity (closure of OPE) and modular invariance. Even though the absence of a tachyon might also be added to the list, the presence of a tachyon in the bosonic string does not indicate any fundamental inconsistency in the theory.<sup>8</sup> Also, for modular invariance, it is sufficient to check associativity of OPE and modular invariance of the one-point amplitude at one-loop [31].

It does not sound an easy job for a string theory to satisfy all these requirements. However, there is a common belief that a world-sheet anomaly (either local or global) always leads to a spacetime anomaly.<sup>9</sup> So, a string theory is likely to be automatically consistent once world-sheet anomalies are removed. If this is true even for curved backgrounds, the most plausible solution to our ghost problem is the possibility 2). This might be related to 1). However, the modular invariance for a string theory in a curved spacetime is a hard problem and not well understood.

### *Closure of OPE*

Unitarity requires closure of OPE, and fusion rules are determined by tensor products of the underlying primaries and by null states in Kac-Moody and Virasoro module. Here we consider constraints on the fusion rule from tensor products of the  $\widetilde{SL}(2, \mathbf{R})$  representations.

Tensor products and the Clebsch-Gordan coefficients of the  $SL(2, \mathbf{R})$  representations including non-unitary ones are discussed in [6]. Since we are dealing only with the unitary representations, the problem is simple and we can use the results in the literature. We have summarized tensor products of the unitary representations in Appendix A. We find that the tensor products are closed if the content of the operators is given by (i) only the highest (or the lowest) discrete series, or (ii) the highest, lowest discrete series and the principal continuous series, so that the addition and subtraction of the  $j$ -values are closed mod  $\mathbf{Z}$ . Once we add the complementary series, we have to include all the other unitary

---

<sup>8</sup>See however Ref. [30], which might imply that the bosonic string does not exist nonperturbatively.

<sup>9</sup> Some works on this theme are as follows: the connection of the modular invariance and spacetime anomalies are discussed in [32] (for the type I) and [33] (for the type II and the heterotic string); the connection of the modular invariance and unitarity are discussed in [32, 34].

series. These are the necessary conditions for the closure of the OPE.

### *Partition function and modular invariance*

From the spectrum in Sec. 3, we get

$$\begin{aligned} L_0 - \tilde{L}_0 &= -nm + N - \tilde{N}, \\ L_0 + \tilde{L}_0 &= \frac{-2j(j+1)}{k-2} + N + \tilde{N} - n \left( \frac{k}{2} \Delta_+^2 n - 2\Delta_+ J_R + m \right). \end{aligned} \quad (6.1)$$

The partition function diverges since the Casimir  $-j(j+1)$ ,  $J_R$  and two integers  $n, m$  can take arbitrarily large or small values. In Minkowski spacetime, we can avoid the divergence of the partition function by the Wick rotation, but we have no analogue in our case. Furthermore, our Kac-Moody module is restricted to the states of the form (3.24), so we have to take this into account in the character calculation.

One resolution to this problem might be to find a subclass of the spectrum and/or to develop an analogue of the Wick rotation so that we get a finite and modular invariant partition function. This might also solve the ghost problem. For compact group manifolds [24], the spectrum is restricted to integrable representations of the Kac-Moody algebra, so that we can get modular invariant partition functions. Fields in non-integrable representations decouple in correlators. However, the argument depend largely upon compactness, so we have to take different strategies for non-compact cases. So far, there is no general argument, but for the  $SL(2, \mathbf{R})$  theory, there are a few attempts[12, 14]. Besides group manifolds, partition functions of string theories on curved spacetime are discussed in [17].

## 6.2 Discrete symmetries

One possibility to consistently truncate the spectrum is further orbifolding besides that with respect to  $\varphi \sim \varphi + 2\pi$ . As we will see, only part of the  $\widetilde{SL}(2, \mathbf{R})$  manifold is necessary to describe three dimensional black holes. Since we have started with the  $\widetilde{SL}(2, \mathbf{R})$  WZW model, the redundant part of the manifold should be divided away by orbifolding. In this subsection, we will discuss the relevant discrete symmetries.

In Appendix A, we see that the  $SL(2, \mathbf{R})$  manifold contains sixteen domains denoted by  $\pm D_i^\pm$  ( $i = 1-4$ ). One correspondence between Region I-III and these domains is

$$\text{Region I} = D_1^+, \quad \text{Region II} = D_2^- \cap (-D_3^+), \quad \text{Region III} = -D_4^-. \quad (6.2)$$

Here we have taken a parametrization in Region II and III slightly different from the one in Sec. 2, but the geometry is the same. Thus we need only the universal covering of the region  $\Omega_1 \equiv D_1^+ \cap D_2^- \cap (-D_3^+) \cap (-D_4^-)$  to get the black hole geometry, as long as we do not consider its maximal extension. Now let us define two transformations by

$$\begin{aligned} T_1 : g &\rightarrow g' = -g, \\ T_2 : g &\rightarrow g' = \mathcal{B}g \quad \text{in } \pm D_{1,2}^\pm, \quad g' = -\mathcal{B}g \quad \text{in } \pm D_{3,4}^\pm, \end{aligned} \quad (6.3)$$

where  $\mathcal{B}$  is given by (A.36) and called Bargmann's automorphism of  $SL(2, \mathbf{R})$ .  $T_{1,2}$  have the properties

$$\begin{aligned} T_1^2 &= T_2^2 = 1, \\ T_1 : \Omega_{1(2)} &\rightarrow -\Omega_{1(2)}, \quad T_2 : \Omega_{1(2)} \rightarrow \Omega_{2(1)}, \end{aligned} \quad (6.4)$$

where  $\Omega_2 = (D_1^- \cap D_2^+ \cap D_3^- \cap D_4^+)$ . Note that  $\pm\Omega_{1,2}$  cover all sixteen domains of  $SL(2, \mathbf{R})$  and have no overlap among them. Moreover we can obtain the black hole geometry from each of the four sets as in Sec. 2. Thus we can divide  $SL(2, \mathbf{R})$  by the  $\mathbf{Z}_2$  symmetries,  $T_1$  and  $T_2$ , in order to drop redundant regions.

There is one more discrete symmetry. This is related to the problem of closed timelike curves. Region I-III or each of  $\pm\Omega_{1,2}$  includes the region  $r^2 < 0$  where closed timelike curves exist [18]. This region corresponds to part of  $-D_4^-$  in  $\Omega_1$  for the rotating case or the whole region for the non-rotating case. Although we have no symmetry to remove this region only, it is possible to drop it together with the region  $(r_+^2 + r_-^2)/2 > r^2 > 0$ . The region  $(r_+^2 + r_-^2)/2 > r^2$  corresponds to  $(-D_3^+) \cap (-D_4^-)$  in  $\Omega_1$ , so we have only to find a symmetry between  $D_1^+ \cap D_2^+$  and  $(-D_3^+) \cap (-D_4^-)$ . The symmetry is easy to find in coordinates  $(\hat{t}, \hat{\varphi}, \hat{r})$ . Let us define a  $\mathbf{Z}_2$  transformation by

$$T_3 : (\hat{t}, \hat{\varphi}, \hat{r}^2 - 1/2) \rightarrow (\hat{\varphi}, \hat{t}, -(\hat{r}^2 - 1/2)). \quad (6.5)$$

Then by recalling that the geometry is given by (2.7), we find that the metric and the antisymmetric tensor are invariant under  $T_3$ . This symmetry maps any point in  $D_1^+ \cap D_2^+$  ( $\hat{r}^2 > 1/2$ ) to one in  $(-D_3^+) \cap (-D_4^-)$  ( $\hat{r}^2 < 1/2$ ) and vice versa. Thus we can truncate both the spectrum and the region with closed timelike curves by the orbifolding with respect to  $T_3$  at the expense of the additional dropped region. Notice that part of  $T_3$ ,  $\hat{r}^2 \rightarrow 1 - \hat{r}^2$  or  $\hat{t} \leftrightarrow \hat{\varphi}$ , has already appeared in the discussion of the T-duality in Sec. 5.

## Acknowledgements

We would like to thank S. Hirano, G. Horowitz, H. Ishikawa, M. Kato, Y. Kazama, Y. Matsuo and N. Sakai for useful discussions. We would especially like to thank T. Oshima for useful discussions on the representation theory of  $SL(2, \mathbf{R})$  and J. Polchinski for useful discussions and comments on the draft of this paper. This work was supported in part by JSPS Research Fellowship for Young Scientists No. 07-4678 and No. 06-4391.

## A Representations of $SL(2, \mathbf{R})$

In this appendix, we briefly summarize the representation theory of  $SL(2, \mathbf{R})$  (and its universal covering group  $\widetilde{SL}(2, \mathbf{R})$ ) and collect its useful properties for discussions in this paper. For a review, see [25] and [35]-[37].



## A.1 $SL(2, \mathbf{R})$

### A.1.1 Preliminary

The group  $SL(2, \mathbf{R})$  is represented by real matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \quad (\text{A.1})$$

It has one-parameter subgroups

$$\Omega_a = \{g_a(t) = e^{-it\tau^a}\}, \quad a = 0, 1, 2, \quad (\text{A.2})$$

where

$$\begin{aligned} \tau^0 &= -\frac{1}{2}\sigma_2 \rightarrow g_0(t) = \begin{pmatrix} \cos t/2 & \sin t/2 \\ -\sin t/2 & \cos t/2 \end{pmatrix}, \\ \tau^1 &= \frac{i}{2}\sigma_1 \rightarrow g_1(t) = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}, \\ \tau^2 &= \frac{i}{2}\sigma_3 \rightarrow g_2(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \end{aligned} \quad (\text{A.3})$$

where  $\sigma_i$  ( $i = 1-3$ ) are the Pauli matrices. In  $\Omega_0$ ,  $g_0(0)$  and  $g_0(4\pi)$  represent the same point and  $g_0(t)$ ,  $t \in [0, 4\pi)$  traces an uncontractable loop in  $SL(2, \mathbf{R})$ . If we unwrap this loop and do not identify  $g_0(0)$  and  $g_0(4\pi)$ , we get the universal covering group  $\widetilde{SL}(2, \mathbf{R})$ .  $\tau^a$  ( $a = 0, 1, 2$ ) have the properties

$$[\tau^a, \tau^b] = i\epsilon^{ab}{}_c \tau^c, \quad \text{Tr}(\tau^a \tau^b) = -\frac{1}{2}\eta^{ab}, \quad (\text{A.4})$$

where  $\eta^{ab} = \text{diag}(-1, 1, 1)$ .  $\tau^a$  form a basis of  $sl(2, \mathbf{R})$ .

$SL(2, \mathbf{R})$  is isomorphic to  $SU(1, 1)$  (and so is  $sl(2, \mathbf{R})$  to  $su(1, 1)$ ). The isomorphism is given by

$$\tilde{g} = T^{-1} g T, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad (\text{A.5})$$

where  $\tilde{g} \in SU(1, 1)$  and  $g \in SL(2, \mathbf{R})$ . Note  $\tilde{g}_0$  is diagonal in  $SU(1, 1)$ , while so is  $g_2$  in  $SL(2, \mathbf{R})$ .

### A.1.2 Parametrization

Any matrix  $g$  of  $SL(2, \mathbf{R})$ , with all its elements being non-zero, can be represented as

$$g = d_1 (-e)^{\epsilon_1} s^{\epsilon_2} p d_2. \quad (\text{A.6})$$

Here,  $\epsilon_{1,2} = 0$  or  $1$ ;  $d_i = \text{diag}(e^{\psi_i/2}, e^{-\psi_i/2})$  ( $i = 1, 2$ );

$$-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{A.7})$$

and  $p$  is one of the following matrices:

$$\begin{aligned} p &= g_1(\theta), & -\infty < \theta < +\infty, \\ p &= g_0(\theta), & -\pi/2 < \theta < +\pi/2. \end{aligned} \quad (\text{A.8})$$

Thus,  $SL(2, \mathbf{R})$  has eight domains given by

$$\begin{aligned} D_1 &= \left\{ A_1 = \begin{pmatrix} e^\phi \cosh \theta/2 & e^\psi \sinh \theta/2 \\ e^{-\psi} \sinh \theta/2 & e^{-\phi} \cosh \theta/2 \end{pmatrix}, -\infty < \theta < +\infty \right\}, \\ D_2 &= \left\{ A_2 = \begin{pmatrix} e^\phi \cos \theta/2 & e^\psi \sin \theta/2 \\ -e^{-\psi} \sin \theta/2 & e^{-\phi} \cos \theta/2 \end{pmatrix}, -\frac{\pi}{2} < \theta < +\frac{\pi}{2} \right\}, \\ D_3 &= \left\{ A_3 = \begin{pmatrix} -e^\phi \sin \theta/2 & e^\psi \cos \theta/2 \\ -e^{-\psi} \cos \theta/2 & -e^{-\phi} \sin \theta/2 \end{pmatrix}, -\frac{\pi}{2} < \theta < +\frac{\pi}{2} \right\}, \\ D_4 &= \left\{ A_4 = \begin{pmatrix} e^\phi \sinh \theta/2 & e^\psi \cosh \theta/2 \\ -e^{-\psi} \cosh \theta/2 & -e^{-\phi} \sinh \theta/2 \end{pmatrix}, -\infty < \theta < +\infty \right\}, \\ -D_i &= \{-A_i\} \quad (i = 1 \sim 4), \end{aligned} \quad (\text{A.9})$$

where  $-\infty < \phi, \psi < +\infty$ . We can further divide these domains according to the sign of  $\theta$ . We denote the domains with positive  $\theta$  by  $\pm D_i^+$  and those with negative  $\theta$  by  $\pm D_i^-$ .

When a matrix element of  $g$  is zero, it is for example written by  $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$ . Taking appropriate limits of  $\pm A_i$  yields such a matrix.

## A.2 Unitary representations

Let us denote the generators of  $sl(2, \mathbf{R})$  by  $J^a$  and consider the basis given by  $I^0 = J^0$  and  $I^\pm = J^1 \pm iJ^2$ . In this basis, the nontrivial commutation relations read

$$[I^0, I^\pm] = \pm I^\pm, \quad [I^+, I^-] = -2I^0. \quad (\text{A.10})$$

This basis is natural from the  $su(1,1)$  point of view because  $I^0$  corresponds to diagonal elements and  $I^\pm$  are regarded as ladder operators as in  $su(2)$ . Using this basis, we can classify all unitary representations of  $sl(2, \mathbf{R})$  and hence those of  $SL(2, \mathbf{R})$  and  $\widetilde{SL}(2, \mathbf{R})$  [35],[25, 10]. There are five classes of the unitary representations of  $sl(2, \mathbf{R})$  which are labeled by the Casimir  $\mathbf{C} = \eta_{ab} J^a J^b$ ,  $I^0$  and a parameter  $m_0 \in [0, 1)$ :

1. Principal continuous series  $T_\chi^P$ : Representations realized in  $\{|j, m\rangle\}$ ,  $m = m_0 + k$ ,  $0 \leq m_0 < 1$ ,  $k \in \mathbf{Z}$  and  $j = -1/2 + i\nu$ ,  $0 < \nu$ .
2. Complementary (Supplementary) series  $T_\chi^C$ : Representations realized in  $\{|j, m\rangle\}$ ,  $m = m_0 + k$ ,  $0 \leq m_0 < 1$ ,  $k \in \mathbf{Z}$ , and  $\min\{-m_0, m_0 - 1\} < j \leq -1/2$ .
3. Highest weight discrete series  $T_j^H$ : Representations realized in  $\{|j, m\rangle\}$ ,  $m = M_{max} - k$ ,  $k \in \mathbf{Z}_{\geq 0}$  and  $j = M_{max} \leq -1/2$  such that  $I^+ |j, j\rangle = 0$ .

4. Lowest weight discrete series  $T_j^L$  : Representations realized in  $\{|j, m\rangle\}$ ,  $m = M_{min} + k$ ,  $k \in \mathbf{Z}_{\geq 0}$  and  $j = -M_{min} \leq -1/2$  such that  $I^- |j, -j\rangle = 0$ .
5. Identity representation : The trivial representation  $| -1, 0\rangle$ .

Here,  $\chi$  is the pair  $(j, m_0)$ ;  $\mathbf{Z}_{\geq 0}$  refers to non-negative integers; and we have denoted the value of  $\mathbf{C}$  by  $-j(j+1)$ . Note that  $j$  need not be real although  $-j(j+1)$  should be and that we can restrict  $j$  to  $\text{Im } j > 0$  for 1) and  $j \leq -1/2$  for the others because  $j$  and  $-(j+1)$  represent the same Casimir.

Unitary representations of  $\widetilde{SL}(2, \mathbf{R})$  are realized in the same space  $\{|j, m\rangle\}$ . For  $SL(2, \mathbf{R})$ , the parameters are further restricted to  $m_0 = 0, 1/2$  in 1),  $m_0 = 0$  in 2) and  $j = (\text{half integers})$  in 3) and 4). We will use the same notations as in  $sl(2, \mathbf{R})$ .

From the harmonic analysis on  $\widetilde{SL}(2, \mathbf{R})$ , a complete basis for the square integrable functions on  $\widetilde{SL}(2, \mathbf{R})$  is given by the matrix elements of the principal continuous series, the highest and lowest weight discrete series.

### A.3 Tensor product

Because we have various unitary representations, the decomposition of tensor products is more complicated than  $SU(2)$ . Basic strategy to get the decomposition is to decompose the tensored representation spaces into the eigenspaces of the Casimir [38, 39]. We are interested in tensor products among  $T_\chi^P$  and  $T_j^{H,L}$ . For  $SL(2, \mathbf{R})$ , the decompositions are given as follows [25, 40] :

- 1) For two discrete series of the same type,

$$T_{j_1}^{L,H} \otimes T_{j_2}^{L,H} = \sum_{n=0}^{\infty} \oplus T_{j_1+j_2-n}^{L,H}. \quad (\text{A.11})$$

- 2) For two discrete series of different types,

$$T_{j_1}^L \otimes T_{j_2}^H = \int_0^\infty T_{(-1/2+i\rho, m_0)}^P d\mu(\rho) \oplus \sum_{j=-m_0-1}^{j_1-j_2} (T_j^L \oplus T_j^H), \quad (\text{A.12})$$

where  $m_0 = j_1 - j_2 \bmod \mathbf{Z}$  and  $d\mu(\rho)$  is a continuous measure. We have assumed  $j_2 \geq j_1$ , but the opposite case is obtained similarly. We remark that  $j \leq -m_0 - 1$  and the identity representation does not appear in the right-hand side [40].<sup>10</sup>

- 3) For a discrete and a principal continuous series,

$$T_{j_1}^{L,H} \otimes T_{(-1/2+i\rho', m'_0)}^P = \int_0^\infty T_{(-1/2+i\rho, m_0)}^P d\mu(\rho) \oplus \sum_{j=-m_0-1}^{-\infty} T_j^{L,H}, \quad (\text{A.13})$$

---

<sup>10</sup> In [41], it is claimed that the identity representation does appear as an exceptional case. In our understanding, they show just the existence of the solution to the recursion equation for the Clebsch-Gordan coefficients.

where  $m_0 = m'_0 + j_1 \bmod \mathbf{Z}$ .

4) For two continuous series,

$$\begin{aligned} & T_{(-1/2+i\rho', m'_0)}^P \otimes T_{(-1/2+i\rho'', m''_0)}^P \\ &= \int_0^\infty T_{(-1/2+i\rho, m_0)}^P d\mu_1(\rho) \oplus \int_0^\infty T_{(-1/2+i\rho, m_0)}^P d\mu_2(\rho) \oplus \sum_{j=-m_0-1}^{-\infty} (T_j^L \oplus T_j^H), \end{aligned} \quad (\text{A.14})$$

where  $m_0 = m'_0 + m''_0 \bmod \mathbf{Z}$ .

The decomposition is determined essentially by local properties of the group as is clear from the consideration of tensor products of  $sl(2, \mathbf{R})$ . Thus the decompositions for  $\widetilde{SL}(2, \mathbf{R})$  are obtained by continuing the value of  $m_0$  and  $j$ .

For completeness, we mention tensor products including the complementary series [38, 39]. The tensor product of a principal and a complementary series, or that of two complementary series is decomposed into principal and discrete series like (A.14). In the latter case, one complementary series may appear additionally. The tensor product of a complementary and a discrete series is similar to that of a principal and a discrete series [39].

The Clebsch-Gordan coefficients have been discussed in [41, 40], [25, 37], [6].

## A.4 Representations in the hyperbolic basis

In Appendix A.2, we have discussed the representations in the basis diagonalizing  $J^0 = I^0$  which is the compact direction of  $SL(2, \mathbf{R})$ . We can also consider bases diagonalizing  $J^2$  or  $J^- = J^0 - J^1$  which are non-compact directions [25], [40], [42]-[45], [4]. The generators  $J^0$ ,  $J^2$  and  $J^-$  are called elliptic, hyperbolic and parabolic respectively. One outstanding feature of non-compact generators is that they have continuous spectra. In the rest of this appendix, we will concentrate on representations in the hyperbolic basis.

In terms of  $J^\pm \equiv J^0 \pm J^1$  and  $J^2$ , the commutation relations (A.4) are given by

$$[J^+, J^-] = -2iJ^2, \quad [J^2, J^\pm] = \pm iJ^\pm. \quad (\text{A.15})$$

The latter equation indicates that the ladder operators  $J^\pm$  change the eigenvalue of  $J^2$  by  $\pm i$ . This seems to contradict the Hermiticity of  $J^2$ . However, this is not the case [42].

In general, the eigenvalue of an Hermite operator with *continuous* spectrum need not be real [46], but for our purpose it is convenient to choose spectrum with real values. Thus, we use the basis given by  $\{|\lambda\rangle\}$ , where  $\lambda$  is the eigenvalue of  $J^2$  and runs through all the real number. For the principal continuous and the complementary series, the eigenvalue of  $J^2$  has multiplicity two. Thus the basis has an index  $\pm$  to distinguish them and is given by  $\{|\lambda\rangle_\pm\}$ . In the remainder of this section, we omit this and the other indices to

specify representations such as  $j, m_0, L$  and  $H$ . In the above basis, an element (a state) of the representation space is given by a “wave packet”

$$|\phi\rangle = \int_{-\infty}^{\infty} d\lambda \phi(\lambda) |\lambda\rangle, \quad \|\phi\|^2 = \int_{-\infty}^{\infty} d\lambda |\phi(\lambda)|^2 < \infty. \quad (\text{A.16})$$

This is analogous to a state in field theory where one uses a plane wave basis in infinite space. Then the generators act on the state as

$$\begin{aligned} J^2 |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda \lambda \phi(\lambda) |\lambda\rangle, \\ J^+ |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_+(\lambda) \phi(\lambda - i) |\lambda\rangle, \\ J^- |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_-(\lambda + i) \phi(\lambda + i) |\lambda\rangle. \end{aligned} \quad (\text{A.17})$$

$f_{\pm}$  play the role of the matrix elements in this basis. From the above action, the commutation rules are realized if

$$f_+(\lambda) f_-(\lambda) - f_-(\lambda + i) f_+(\lambda + i) = -2i\lambda. \quad (\text{A.18})$$

An eigenstate  $|\lambda'\rangle$  is obtained in the limit  $\phi(\lambda) \rightarrow \delta(\lambda - \lambda')$ .

It is possible to introduce  $|\lambda \pm i\rangle$  and write the action of the generators as

$$\begin{aligned} J^+ |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_+(\lambda + i) \phi(\lambda) |\lambda + i\rangle, \\ J^- |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_-(\lambda) \phi(\lambda) |\lambda - i\rangle, \\ J^+ |\lambda\rangle &= f_+(\lambda + i) |\lambda + i\rangle, \quad J^- |\lambda\rangle = f_-(\lambda) |\lambda - i\rangle. \end{aligned} \quad (\text{A.19})$$

In this way, we can formally consider eigenstates  $|\lambda \pm i\rangle$ . However, we should always understand them in the sense of (A.17). Note that  $|\lambda \pm i\rangle$  can be “expanded” by the original basis  $\{|\lambda\rangle\}$ , where  $\lambda \in \mathbf{R}$ .

Now let us consider the matrix elements of  $J^{\pm}$ . In the elliptic basis, we get the matrix elements of  $I^{\pm}$  by evaluating the commutation relation  $[I^+, I^-] = -2I^0$  between eigenstates of  $I^0$ . In the hyperbolic basis, this method dose not work because  $(J^{\pm})^{\dagger} = J^{\pm}$ . First, note that the Casimir takes the form

$$\begin{aligned} \mathbf{C} &= \eta_{ab} J^a J^b \\ &= J^2(J^2 + i) - J^- J^+ = J^2(J^2 - i) - J^+ J^-. \end{aligned} \quad (\text{A.20})$$

The condition (A.18) has the solution  $f_+(\lambda) f_-(\lambda) = \lambda(\lambda - i) - c$ , where  $c$  is a constant. Moreover, evaluating the Casimir on an eigenstate of  $J^2$  leads to  $c = -j(j + 1)$ , i.e.,

$$f_+(\lambda) f_-(\lambda) = \lambda(\lambda - i) + j(j + 1) \equiv d^2(j, \lambda - i). \quad (\text{A.21})$$

We cannot determine  $f_+$  or  $f_-$  separately without additional conditions. Consequently, we find that

$$J^+ J^- |j; \lambda\rangle = d^2(j, \lambda - i) |j; \lambda\rangle, \quad J^- J^+ |j; \lambda\rangle = d^2(j, \lambda) |j; \lambda\rangle. \quad (\text{A.22})$$

Note that  $d^2(j, \lambda - i) = \overline{d^2(j, \lambda)}$ .

In the elliptic basis, the commutation relations are given by (A.10) and the Casimir is by

$$\mathbf{C} = -I^0(I^0 + 1) + I^- I^+ = -I^0(I^0 - 1) + I^+ I^-. \quad (\text{A.23})$$

From them, we find that the actions of  $I^\pm$  on an eigenstate of  $\mathbf{C}$  and  $I^0$  are

$$I^- I^+ |j; m\rangle = \tilde{d}^2(j, m) |j; m\rangle, \quad I^+ I^- |j; m\rangle = \tilde{d}^2(j, m - 1) |j; m\rangle, \quad (\text{A.24})$$

where  $\tilde{d}^2(j, m) = -j(j + 1) + m(m + 1)$ . So, we see that (A.10), (A.23) and (A.24) are related to the corresponding equations in the hyperbolic basis by “analytic continuation”  $I^\pm \rightarrow iJ^\pm$  and  $I^0 \rightarrow -iJ^2$  [42].

## A.5 Matrix elements

By explicit realization of the representations in spaces of function, we can calculate the matrix elements of  $SL(2, \mathbf{R})$ . Here we consider the matrix elements in the hyperbolic basis [25], [44, 45], [4].

First, let us discuss the principal continuous series  $T_\chi^P$  of  $SL(2, \mathbf{R})$ . This representation is realized in a space of functions on a real axis,  $\mathcal{I}_\chi$ . The action of the group element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$  and the inner product are given by

$$(T_\chi^P(g)f)(x) = |bx + d|^{2j} \text{sign}^{2m_0}(bx + d) f\left(\frac{ax + c}{bx + d}\right), \quad (\text{A.25})$$

$$(f_1(x), f_2(x)) = \int_{-\infty}^{\infty} dx \overline{f_1(x)} f_2(x). \quad (\text{A.26})$$

Then we find that

$$\psi_{\lambda\pm}^\chi(x) \equiv \frac{1}{\sqrt{2\pi}} x^{-i\lambda+j} \theta(\pm x), \quad \lambda \in \mathbf{R}, \quad (\text{A.27})$$

form an orthonormal basis diagonalizing the action of  $J^2$ , namely

$$(\psi_{\lambda\epsilon}^\chi(x), \psi_{\mu\epsilon'}^\chi(x)) = \delta_{\epsilon\epsilon'} \delta(\lambda - \mu), \quad (\text{A.28})$$

$$[T_\chi^P(g_2(t)) \psi_{\lambda\pm}^\chi](x) = e^{-it\lambda} \psi_{\lambda\pm}^\chi(x), \quad g_2(t) \in \Omega_2, \quad (\text{A.29})$$

where  $\epsilon, \epsilon' = \pm$ .  $\psi_{\lambda\pm}^\chi$  correspond to  $|\lambda\rangle_\pm$  in the previous subsection and are not elements in  $\mathcal{I}_\chi$ .

We can calculate the matrix elements in the basis (A.27) using (A.25) and (A.26). For example, for  $t > 0$  we have

$${}^P D_{\lambda+, \lambda'+}^{\chi}(g_1(t)) = \frac{1}{2\pi} B(\mu, -\mu' - 2j) \frac{\cosh^{2j+\mu+\mu'} t/2}{\sinh^{\mu+\mu'} t/2} \times F(\mu, \mu'; -2j; -\sinh^{-2} t/2), \quad (\text{A.30})$$

$${}^P D_{\lambda-, \lambda'-}^{\chi}(g_1(t)) = \frac{1}{2\pi} B(1 - \mu', \mu' - 1 + 2(j+1)) \frac{\cosh^{2j+\mu+\mu'} t/2}{\sinh^{4j+2+\mu+\mu'} t/2} \times F(\mu + 2j + 1, \mu' + 2j + 1; 2j + 2; -\sinh^{-2} t/2), \quad (\text{A.31})$$

$${}^P D_{\lambda\epsilon, \lambda'\epsilon'}^{\chi}(g_2(t)) = e^{-it\lambda} \delta_{\epsilon\epsilon'} \delta(\lambda - \lambda'), \quad (\text{A.32})$$

where  $\mu^{(\prime)} = i\lambda^{(\prime)} - j$ .  $F$  and  $B$  are the hypergeometric function and the Euler beta function respectively. For  $g_1(t)$ ,  ${}^P D_{\lambda-, \lambda'+}^{\chi}$  is given by a linear combination of (A.30) and (A.31), and  ${}^P D_{\lambda+, \lambda'-}^{\chi}$  vanishes.

The matrix elements for the complementary series are obtained by analytically continuing the value of  $j$  [44].

Let us turn to the discrete series  $T_j^L$ . This is realized in a space of analytic functions on  $\mathbf{C}_+$  (the upper half-plane). (This can also be embedded in the principal continuous series.) The action of  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$  and the inner product are given by <sup>11</sup>

$$(T_j^L(g)f)(w) = (bw + d)^{2j} f\left(\frac{aw + c}{bw + d}\right), \quad (\text{A.33})$$

$$(f_1(w), f_2(w)) = \frac{i}{2\Gamma(-2j-1)} \int_{\mathbf{C}_+} dw d\bar{w} y^{-2j-2} \overline{f_1(w)} f_2(w), \quad (\text{A.34})$$

where  $w = x + iy$  and  $dw d\bar{w} = -2i dx dy$ . We then find that

$$\varphi_{\lambda}^j(w) = \frac{1}{2^{(j+1)\pi}} e^{-\lambda\pi/2} \Gamma(-i\lambda - j) w^{-i\lambda+j}, \quad \lambda \in \mathbf{R}, \quad (\text{A.35})$$

form an orthonormal basis diagonalizing  $J^2$ . Thus similarly to the previous case (or using the fact that  $f(w)$  is determined by its values on the semi-axis  $w = iy$  ( $y > 0$ )), we can calculate the matrix elements.  ${}^L D_{\lambda, \lambda'}^j(g_1(t))$  is the same up to a numerical factor as (A.30) and  ${}^L D_{\lambda, \lambda'}^j(g_2(t))$  is given by (A.32) without  $\delta_{\epsilon\epsilon'}$ .

For the highest weight series  $T_j^H$ , we can get the matrix elements from the lowest weight series. By utilizing an automorphism of  $SL(2, \mathbf{R})$  called Bargmann's automorphism of  $SL(2, \mathbf{R})$

$$\mathcal{B}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad (\text{A.36})$$

---

<sup>11</sup> $j = -1/2$  case needs special treatment, but the matrix elements take the same forms as in  $j < -1/2$  cases [35, 44].

the matrix elements of the highest weight series are given by [25, 44]

$${}^H D_{\lambda, \lambda'}^j(g) = {}^L D_{\lambda, \lambda'}^j(\mathcal{B}g). \quad (\text{A.37})$$

All the matrix elements satisfy the differential equation

$$[\Delta - j(j+1)] D_{\lambda, \lambda'}^{j(\chi)}(g) = 0, \quad (\text{A.38})$$

where  $\Delta$  is the Laplace operator on  $SL(2, \mathbf{R})$  and they are characterized essentially by local properties of  $SL(2, \mathbf{R})$ . Hence, the matrix elements of  $\widetilde{SL}(2, \mathbf{R})$  are obtained by continuing the values of  $j$  and  $m_0$ .

## B Decomposition of the Kac-Moody module

The Clebsch-Gordan decomposition similar to  $su(2)$  holds for  $sl(2, \mathbf{R})$  ( $su(1, 1)$ ) in the elliptic basis [9]. Their argument is valid in the hyperbolic basis as well with a slight modification.

Let  $V^a$  be a vector operator, i.e.,

$$[J_0^a, V^b] = i\epsilon^{ab}_c V^c, \quad (\text{B.1})$$

and  $|j; \lambda\rangle$  be an eigenstate of  $\mathbf{C}$  and  $J^2$ . An example is  $V^a = J_{-1}^a$ .  $|j; \lambda\rangle$  need not be a base state of the Kac-Moody module. Let us consider states

$$V^+ J_0^- |j; \lambda\rangle, \quad V^- J_0^+ |j; \lambda\rangle, \quad V^2 |j; \lambda\rangle. \quad (\text{B.2})$$

In the hyperbolic basis, these states do not vanish in any unitary representation. From (A.20), the matrix elements of the Casimir with respect to these states are

$$\mathbf{C} = \begin{pmatrix} c + 2i\lambda & 0 & i \\ 0 & c - 2i\lambda & -i \\ -2id^2(j, \lambda - i) & 2id^2(j, \lambda) & c - 2 \end{pmatrix}, \quad \text{where } c = -j(j+1). \quad (\text{B.3})$$

The trace and determinant in this subspace are given by

$$\text{Tr } \mathbf{C} = 3c - 2, \quad \det \mathbf{C} = c^2(c + 2). \quad (\text{B.4})$$

It is easy to see that the state  $(1, 1, -2\lambda)$  is an eigenvector with the Casimir  $\mathbf{C} = -j(j+1)$ . Then, the other eigenvalues are  $-j(j-1)$  and  $-(j+1)(j+2)$ . Therefore, the states in (B.2) are decomposed into the  $sl(2, \mathbf{R})$  representations with  $j$ -values  $j$  and  $j \pm 1$ . The corresponding eigenvectors  $\psi_j$  and  $\psi_{j \pm 1}$  are given by

$$\begin{aligned} \psi_j &= (1, 1, -2\lambda), \\ \psi_{j-1} &= (-(j+i\lambda), j-i\lambda, 2i(j^2+\lambda^2)), \\ \psi_{j+1} &= (j+1-i\lambda, -(j+1+i\lambda), 2i((j+1)^2+\lambda^2)). \end{aligned} \quad (\text{B.5})$$



Note  $\psi_{j+1}$  is obtained from  $\psi_{j-1}$  by the replacement  $j \rightarrow -j - 1$ .

It is useful to remark on the norm of states. Consider representations where the Casimir operator is Hermitian. The representations need not be unitary. Let  $|\Psi_1\rangle$  and  $|\Psi_2\rangle$  be eigenstates with the Casimir values  $c_1$  and  $c_2$  respectively. Then by evaluating the matrix element  $(\Psi_1, \mathbf{C}\Psi_2) = (\mathbf{C}\Psi_1, \Psi_2)$ , we get

$$(\bar{c}_1 - c_2) \langle \Psi_1 | \Psi_2 \rangle = 0. \quad (\text{B.6})$$

Therefore, for complex  $c_1$  and  $c_2$ , the norm vanishes when  $c_1 = c_2$ . It can be non-zero only when  $c_1$  and  $c_2$  are complex conjugate. Since extremal states constructed on a principal continuous series have complex Casimir values (see Sec. 4), they are physical states with zero norm. On the other hand,  $\langle E_N^+ | E_N^- \rangle$  can be non-zero because their Casimir values are complex conjugate.

## References

- [1] E. Witten, Phys. Rev. **D44** (1991) 314.
- [2] G. Mandal, A. Sengupta and S. Wadia, Mod. Phys. Lett. **A6** (1991) 1685.
- [3] A. Giveon, Mod. Phys. Lett. **31A** (1991) 2843.
- [4] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B371** (1992) 269.
- [5] J. Distler and P. Nelson, Nucl. Phys. **B374** (1992) 123.
- [6] S. Chaudhuri and J. Lykken, Nucl. Phys. **B396** (1993) 270.
- [7] K. Itoh, H. Kunitomo, N. Ohta and M. Sakaguchi, Phys. Rev. **D48** (1993) 3793.
- [8] P. Goddard and C.B. Thorn, Phys. Lett. **40B**, 235 (1972);  
R. Brower, Phys. Rev. **D6**, 1655, (1972);  
M. Kato and K. Ogawa, Nucl. Phys. **B212**, 443 (1983);  
C.B. Thorn, Nucl. Phys. **B286** (1987) 61;  
D. Ghoshal and S. Mukherji, Mod. Phys. Lett. **A6** (1991) 939.
- [9] J. Balog, L. O’Raifeartaigh, P. Forgács and A. Wipf, Nucl. Phys. **B325** (1989) 225.
- [10] L. Dixon, J. Lykken and M. Peskin, Nucl. Phys. **B325** (1989) 329.
- [11] P. Petropoulos, Phys. Lett. **B236** (1990) 151;  
S. Hwang, Nucl. Phys. **B354** (1991) 100.
- [12] M. Henningson, S. Hwang, P. Roberts and B. Sundborg, Phys. Lett. **B267** (1991) 350.
- [13] P. Griffin and O. Hernandez, Nucl. Phys. **B313** (1991) 287.
- [14] K. Huitu, Phys. Lett. **B313** (1993) 75.

- [15] I. Bars and D. Nemeschansky, Nucl. Phys. **B348** (1991) 89.
- [16] I. Bars, Phys. Rev. **D53** (1996) 3308.
- [17] J.G. Russo and A.A. Tseytlin, Nucl. Phys. **B449** (1995) 91.
- [18] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69** (1992) 1849;  
M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. **D48** (1993) 1506.
- [19] G.T. Horowitz and D.L. Welch, Phys. Rev. Lett. **71** (1993) 328.
- [20] N. Kaloper, Phys. Rev. **D48** (1993) 2598.
- [21] S. Carlip, Class. Quant. Grav. **12** (1995) 2853.
- [22] I. Bars and K. Sfetsos, Phys. Rev. **D46** (1992) 4510;  
A.A. Tseytlin, Nucl. Phys. **B399** (1993) 601; **B411** (1994) 509.
- [23] S.B. Giddings, J. Polchinski and A. Strominger, Phys. Rev. **D48** (1993) 5784.
- [24] D. Gepner and E. Witten, Nucl. Phys. **B278** (1986) 493.
- [25] N.Ja. Vilenkin and A.U. Klimyk, *Representation of Lie Groups and Special Functions*,  
(Kluwer Academic Publishers, Dordrecht, 1991).
- [26] K. Ghoroku and A.L. Larsen, Phys. Lett. **B328** (1994) 28;  
M. Natsuume, N. Sakai and M. Sato, Mod. Phys. Lett. **A11** (1996) 1467.
- [27] I. Ichinose and Y. Satoh, Nucl. Phys. **B447** (1995) 340.
- [28] T. Buscher, Phys. Lett. **B194** (1987) 59; **B201** (1988) 466.
- [29] M. Roček and E. Verlinde, Nucl. Phys. **B373** (1992) 630;  
A. Giveon and M. Roček, Nucl. Phys. **B380** (1992) 128.
- [30] J. Polchinski, Phys. Rev. Lett. **74** (1995) 638.
- [31] H. Sonoda, Nucl. Phys. **B311** (1988) 401; 417.
- [32] J. Polchinski and Y. Cai, Nucl. Phys. **B296** (1988) 91.
- [33] A.N. Schellekens and N.P. Warner, Phys. Lett. **177B** (1986) 317; **178B** (1986) 339;  
Nucl. Phys. **B287** (1987) 317.
- [34] J. Polchinski, Nucl. Phys. **B307** (1988) 61; *Joe's Big Book of String* (Cambridge, to appear).
- [35] V. Barmann, Ann. Math. **48** (1947) 568.
- [36] P.J. Sally, Jr., Mem. Am. Math. Soc. **69** (1967).
- [37] B.G. Wybourne, *Classical groups for physicists* (Wiley, New York, 1974).
- [38] L. Pukanszky, Trans. Am. Math. Soc. **100** (1961) 116.

- [39] H. Neunhöffer, Sitzungsber. Heidelb. Akad. Wiss. Math. Natur. Kl. **3** (1978) 173.
- [40] N. Mukunda and B. Radhakrishnan, J. Math. Phys. **15** (1974) 1320; 1332; 1643; 1656.
- [41] W.I. Holeman, III and L.C. Biedenharn, Jr., Ann. Phys. (NY) **39** (1966) 1; **47** (1968) 205.
- [42] J.G. Kuriyan, N. Mukunda and E.C.G. Sudarshan, J. Math. Phys. **9** (1968) 2100.
- [43] G. Lindblad and B. Nagel, Ann. Inst. Henri Poincaré **XIII** (1970) 27.
- [44] N. Mukunda, J. Math. Phys. **10** (1969) 2086; 2092.
- [45] A.O. Barut and E.C. Phillips, Commun. Math. Phys. **8** (1968) 52.
- [46] H. Arisue, T. Fujiwara, T. Inoue and K. Ogawa, J. Math. Phys. **22** (1981) 2055.